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Speculative Dynamics ¹

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Abstract

We develop a method for solving for equilibrium outcomes in stationary strategic settings in which speculators are informationally large and understand how their actions affect the information content of prices. This allows us to characterize speculation by institutional investors who receive private long-lived information on a recurring basis, and trade strategically. When the underlying asset value process has a stationary autoregressive structure, we develop a contraction mapping argument to solve for the stationary linear equilibrium. We derive analytically and numerically how the characteristics of private information—its quantity, persistence and correlation, and division among speculators—affect trading profits, pricing and trading strategies. Our central finding is that what matters for equilibrium outcomes are the most recent signals that speculators receive. Speculators trade so much more aggressively on new information than old that the bulk of their profits come from their two or three most recent private signals. Trading on past prices drops off faster yet; effectively only the most recent price matters.

Keywords: speculation, market microstructure finance, forecasting-the-forecasts, frequency domain, stationary linear equilibrium

JEL: E3, D4, G1, G12

1 Introduction

This paper develops a method for solving for equilibrium outcomes in stationary strategic settings in which agents are informationally large and understand how their actions affect the information content of prices. We use the method to characterize speculative trade and stock price dynamics when speculators acquire private, long-lived information on a recurring basis, and trade strategically.

Existing dynamic models of speculative trade are effectively models of corporate insiders who receive advance information about earnings: speculators acquires information at known dates, and there is a well-defined earnings announcement date at which point everyone becomes fully informed (see Back, Cao and Willard 2000, Holden and Subrahmanyam 1992, Foster and Viswanathan 1994, 1996, Bernhardt and Miao 2004). The researchers then derive how speculators trade over the interim on the information that each has.

This paper considers a very different type of speculation. We model the speculative trade of large investors such as Warren Buffett, or institutions such as Goldman Sachs. Their accumulated expertise, continual research and superior ability to numerically process that information permit them to assess the valuation consequences of information better, and thereby earn vast profits from proprietary equity trades on a consistent basis. Information arrival is ongoing, and these expert speculators understand that they can evaluate past, current and future information better than most other agents. They also understand how their trades affect prices and convey information to others; and that prices contain information about the signals of other speculators that they, themselves, can use. As information about past signals leaks out through price, their value is reduced, but not to zero.

In this context, the problem for speculators becomes: How do they combine current and past private information together with the information in current and past stock prices to determine how intensively to trade on each piece of information that they possess? In particular, how do speculators weigh new information relative to old? These questions cannot even be posed in the models of speculation cited above, where each speculator receives private information only once.

This speculation by institutional investors that we model is a pervasive and unceasing feature of stock markets. However, to address such speculation, we must confront the analytical challenges associated with dealing with strategic interaction with recurring arrival of long-lived information: traders should use the *entire* history of their private signals to process the information in the history of equilibrium prices, and then determine how intensively to trade on each signal and each price taking into account the impacts of their trades on the pricing by the market maker, and, via price, the trades of other speculators¹.

This paper develops a method to solve for equilibrium outcomes in such a setting. We then answer questions such as: How do speculators trade on new information versus old? On private signals versus prices? How does competition among speculators affect trading and price dynamics? Does increased competition

¹This gives rise to a forecasting-the-forecasts of others issues in a heterogeneous agent framework so that traders' forecasts differ conditional on prices—see Townsend (1985), Pearlman and Sargent (2005) where these issues do not arise and Malinova and Smith (2003) who investigates forecasting-the-forecasts in a non-strategic setting.

cause prices to reveal more information? Information about recent innovations? Past innovations? How do the characteristics of private information—its quantity, correlation and persistence—affect trading strategies, pricing, profit and information transmission?

The central message from our analysis is that what matters for equilibrium outcomes are the most recent signals that speculators receive. We find that speculators trade more aggressively on new information than old—to such an extent that they earn the bulk of their profits from their two or three most recent private signals. Trading on past prices drops off faster yet; effectively only the most recent price matters.

We analyze an environment in which the asset value evolves according to an AR stochastic process, and investors privately observe innovations to the asset’s value. We develop an iterative best-response mapping of speculators to the conjectured trading strategies of other speculators. We then develop a novel contraction mapping argument to derive the existence of a stationary linear equilibrium. We prove that the associated equilibrium trading strategies are given by infinite sums of AR(1) functions of their private signals and prices. We then provide tight analytical characterizations of how trading strategies, prices and profits are affected by the amounts of private information and liquidity trade in the economy. Finally, we use the iterative best-response algorithm to characterize numerically how equilibrium outcomes—trading strategies, pricing, volume, information revealed, profits—vary with the primitives that describe the environment.

Our key initial insight is that if the underlying asset values are stationary, then equilibrium strategies, though complicated, will be stationary linear functions of the history of private signals and prices. It follows that the linear trading strategy that maximizes a trader’s expected profits conditionally, also maximizes expected profits unconditionally. We exploit this equivalence and solve this unconditional expected profit-maximization problem, using variational methods to find the optimal trading *functions* (Whiteman 1985). Futia (1982) uses these methods to solve for equilibrium outcomes in a standard noisy REE setting with non-strategic traders. Effectively, we extend these methods to strategic settings.

To prove that a stationary linear equilibrium exists, we construct an iterative best-response mapping, and show that it converges. Specifically, we conjecture that speculators adopt linear trading strategy functions on the history of their private signals. We then solve for the implied pricing and trading on the information in prices. Finally, we assume that each speculator believes that other speculators trade according to this conjecture, and solve for the best response. We iterate on this best-response mapping and construct an indirect contraction mapping argument to prove that there is a fixed point, which corresponds to the equilibrium.²

We then characterize equilibrium outcomes. We prove that an agent’s net total order is equal to his forecast of the error in the market maker’s forecast of his trade on private information. That is, from an agent’s trade on private information, he subtracts off its projection onto the history of prices—he subtracts off the market maker’s forecast of his trade on private signals. We further prove that both the private and public information trading strategy components are infinite sums of AR(1) terms, and that there is no

²Seiler and Taub (2008) establish the robustness of this equilibrium by modifying our methods to apply to a multi-asset setting where there is both slight correlation between assets and slight persistence in asset values.

simpler representation: the best response to a trading strategy with k AR(1) terms, is incrementally more sophisticated, with $k + 1$ terms. We next prove that trading intensities decay faster than do the impacts of innovations to the valuation process. Continuing, we prove that in contrast to the complicated structure of trades, equilibrium pricing takes a simple form: the market maker unwinds the complicated autoregressive structure of the order flow so that the price process has the same AR(1) structure as the asset value process.

We also show how the amounts of private information and liquidity trade affect trading strategies, proving that the weights on private signals are proportional to $\frac{\sigma_u}{\sigma_e}$, the ratio of the standard deviation of liquidity trade to that of signals. It follows that pricing is inversely proportional to $\frac{\sigma_u}{\sigma_e}$ informed profits are proportional to $\sigma_u \sigma_e$. Thus, we prove that properties of simpler models of informed trade such as Kyle (1985) extend.

Lastly, we derive the quantitative impact of the degree of competition between speculators and the persistence and correlation of private information on the intertemporal structure of (a) trading intensities, (b) speculator profit, (c) pricing, and (d) information content of prices. This analysis shows that even when signals are uncorrelated across speculators and the value process is nearly a random walk, speculators earn most of their profits from their two or three most recent private signals. In part, this reflects that past trading reveals information about older signals, but trading intensities on older signals also drop off far faster than do the contributions of the innovations to the value process itself. What drives the reduced trading on older signals is the greater negative correlation in older signals conditional on the information in prices. Trading on past prices drops off faster yet; effectively only the information in the most recent price matters.

We find that both dividing information more finely among speculators and raising signal correlations tilt trading intensities, raising trading on more recent information and reducing trading on older information. However, increased competition has remarkably slight impacts on aggregate speculator profits unless signals are highly correlated. That is, while competition and correlation have complicated impacts on *how* speculators trade, the aggregate consequences of competition are typically *small*. We also find that unless the signal correlation is high, greater competition reduces the market maker's forecast errors about recent innovations, but raises forecast errors on older information, with the surprising result that the total forecast error is *raised*.

Past researchers have focused on simpler settings. The seminal paper, Kyle (1985), considers a single speculator who learns the asset's terminal value at the beginning of the trading game. Back and Pedersen (1998) extend this analysis to allow the speculator to receive information over time, but strategic interactions between agents remain absent.³ Back, Cao and Willard (2000), Holden and Subrahmanyam (1992), and Foster and Viswanathan (1994, 1996) introduce multiple agents who receive symmetrically-distributed signals, but these papers assume that information arrives only at date zero. The equilibrium dynamics of our model are very different from these models. In particular, in our model, trading intensities drop off

³Chau and Vayanos (2008) consider a single speculator in a stationary environment with repeated information arrival, where the market maker sees both order flow and dividends, and dividends are partially driven by the private signal process. The stochastic process of dividends is structured so that in the continuous time limit, the relative contribution of the privately-observed process to value vanishes, so that the speculator can increase trade intensity without reducing market depth: his intense trading causes all private information to be incorporated into price arbitrarily quickly, but he still earns positive profits.

sharply at initial lags, but the declines level off at higher lags. In sharp contrast, in standard models, trading intensity on the same piece of information initially drops very slowly over time, but as the timing of public revelation of information draws near, trading intensities plummet, as the price impact of order flow goes to infinity. Qualitatively, with recurring information arrival, our economy never reaches the “late” stages of models where information arrives only once and the timing of public release of information is known.

Boulatov and Livdan (2007) modify the continuous time framework of Back et al. by introducing market closures and allowing for new signal arrivals about the fixed value of the asset during closures. Bernhardt and Miao (2004) extend these analyses to arbitrary finite horizon environments in which agents can acquire distinct signals at different dates of varying qualities and correlations. Bernhardt and Miao prove that trading strategies are linear functions of unrevealed private information. However, they cannot provide more specific analytical characterizations, and their quantitative characterizations are limited to three periods, as the conditional variance-covariance matrix of private information blows up in their non-stationary setting.

A second approach has been to assume that private information is short-lived: as new private information arrives, old information is revealed to the market (Admati and Pfleiderer 1988). Then agents, when deciding how much to trade, need not trade off current for future profit. A third approach has been to build noisy REE models to characterize volume and price dynamics when agents are price takers (Wang 1994, He and Wang 1995, Malinova and Smith 2003, Makarov and Rytchkov 2007). Assuming that agents are informationally small and ignore the price impact of their trades circumvents individual strategic behavior, as agents do not have to trade off profit-taking against information release. But, allowing for strategic informed trade is important—in practice, informed agents for a given stock are few in number, and these speculators are typically institutional traders who understand that their significant trading has price impacts that they should and do anticipate and internalize.

These models of speculation share the dealership market structure of Kyle (1985). However, no real world analogue of this market design exists. In sharp contrast, in Kyle (1989) agents submit demand schedules that detail how much they want to trade at each price, and the equilibrium price clears the market, equating supply to demand. This institutional structure is used to determine opening prices on most exchanges, and is used exclusively for small stocks on exchanges such as the Paris Bourse. We provide a dynamic analysis of demand-submission markets, exploiting the fact that equilibrium outcomes correspond to those in a competitive dealership market where speculators know the market-clearing price when submitting their orders (Bernhardt and Taub 2006). That is, a demand submission market is strategically equivalent to a noisy rational expectations setting in which speculators internalize the fact that they are informationally large.

We conclude the introduction with the caveat that our analytical approach fully exploits the stationarity of our underlying economy—in particular, information arrival is constant over time, and the frequency of trading opportunities corresponds to the frequency of arrival rates of new information. In practice, speculators such as Goldman Sachs typically have greater informational advantages at some times than others—for example, they may be more able to process information around earnings announcements than

other agents—and the arrival rate of trading opportunities exceeds that of information. In all likelihood, the real world is “somewhere in between” the standard settings considered by the literature, where there is a known terminal date after which all parties are symmetrically informed, and our setting, where the expected informational advantage of speculators is constant over time. In particular, because Goldman Sach’s informational advantage hinges on its superior ability to crunch numbers (and not on its access to specific earnings information), their informational advantage does not vary so much with the earnings cycle. That is, in practice, the informational advantage of speculators never vanishes (in contrast to standard models), even though it varies a little over the earnings cycle in predictable ways (in contrast to our model).

We next present the economic environment. Section 3 analyzes trading strategies and pricing. Section 4 develops the contraction mapping argument used to prove existence of the equilibrium. Section 5 analyzes properties of equilibrium strategies. Section 6 quantitatively characterizes equilibrium outcomes. A conclusion follows. All proofs are in Appendix A and B. We provide a guide to the frequency domain methods that we use in an online appendix, <http://econ.uiuc.edu/~bart/frequencydomainmethods>.

2 The Model

N risk-neutral informed speculators and exogenous liquidity traders trade claims to a firm over time in a market made by risk-neutral competitive, uninformed market makers. Speculators share a common discount factor $\bar{\beta} < 1$. At the end of each date t there is a probability π that the firm is “liquidated”. This liquidation date is best interpreted as a date at which all private information becomes public, in which case speculators cease to be able to extract positive profits from the past private information that they have acquired. So with probability $1 - \pi$ then, the firm continues in its private state on to date $t + 1$. The firm’s liquidation value evolves stochastically over time according to the sum of N first-order autoregressive (AR(1)) processes,

$$\bar{v}_t = \rho \bar{v}_{t-1} + \sum_{j=1}^N \bar{e}_{jt},$$

where \bar{v}_0 is public information, $\rho \in (0, [\bar{\beta}(1-\pi)]^{-1/2}]$, and each value innovation $\bar{e}_{jt} \sim N(0, \frac{\bar{\sigma}^2}{N})$, $j = 1, \dots, N$, is independently and identically distributed across the N processes and time. Our numerical analysis reveals how signal correlation affects outcomes. The AR(1) formulation allows for a rich class of earnings processes; the greater is ρ , the more persistent is the contribution of a date t innovation $\bar{e}_t \equiv \sum_{j=1}^N \bar{e}_{jt}$ to future earnings.⁴

Information. Speculator i has private information about the \bar{e}_i process: at date t , only speculator i knows the date- t history of those innovations, $(\bar{e}_{it}, \bar{e}_{it-1}, \bar{e}_{it-2}, \dots, \bar{e}_{i0})$.⁵ In addition, each speculator knows the

⁴We note that the range of the persistence parameter ρ includes unity, corresponding to the value process following a random walk; and indeed, our theoretical analysis extends to $\rho > 1$, provided that $[\bar{\beta}(1-\pi)]^{-1/2} > \rho$, allowing for a nonstationary evolution of the value process. To maintain tractability however, we focus on $\rho < 1$ in our numerical analysis.

⁵One can interpret date $t = -1$ as the previous date at which all private information of speculators was made public; our framework accommodates “starting over” with new arrival of private information.

date- t history of prices, *including* the date- t price at which orders will be executed. As we noted earlier, equilibrium outcomes correspond to those in a demand submission market.

Pricing. Let x_{it} be speculator i 's order at date t , and let $X_t = \sum_{j=1}^N x_{jt}$ be the total speculative trade. In addition to trade from speculators, there is liquidity trade of u_t . Liquidity trade is independently and identically normally distributed each period, $u_t \sim N(0, \sigma_u^2)$, and is uncorrelated with the asset value process. Total net order flow at date t is $X_t + u_t$ and $\mathbf{\Omega}_{t-1} \equiv (X_{t-1} + u_{t-1}, \dots, X_1 + u_1, \bar{v}_0)$ is the order flow history. The competitive market makers set price equal to the expected value of the asset given this date- t public information, i.e.,

$$p_t = E \left[\sum_{\tau=t}^{\infty} \pi(1-\pi)^{\tau-t} \bar{\beta}^{\tau-t} \bar{v}_{\tau} \mid X_t + u_t, \mathbf{\Omega}_{t-1} \right],$$

where $\pi(1-\pi)^{\tau-t}$ is the probability the firm is liquidated at the end of date $\tau \geq t$ after date τ trading given that it has not been liquidated prior to date t . We focus on equilibria in which market makers set prices that are linear functions of the order flow history. As a result, knowing the history of prices is equivalent to knowing the history of order flows. Thus, at date t , trader i knows the earnings innovation history, $\bar{\mathbf{e}}_{it} \equiv (\bar{e}_{it}, \bar{e}_{it-1}, \dots, \bar{e}_{i1})$, his past orders, $\mathbf{x}_{it} \equiv (x_{it}, x_{it-1}, \dots, x_{i1})$, and the order flow history, $\mathbf{\Omega}_t$.

Speculator Optimization. Consider speculator i 's perspective at some date t given that the firm has not yet been liquidated. Were the firm liquidated at future date τ , i 's net signed position at liquidation would be $\sum_{s \leq \tau} x_{is}$, and the date t value of this position would be $\bar{\beta}^{\tau-t} \bar{v}_{\tau} \sum_{s \leq \tau} x_{is}$. The date t cost of this position would be $\sum_{s \leq \tau} \bar{\beta}^{t-s} p_s x_{is}$, because i 's order x_{is} at date s was executed at price p_s , and from a date t perspective is discounted by $\bar{\beta}^{s-t}$. Thus, if the firm is liquidated at date τ , speculator i 's trading profits equal the difference between the value of his position and its cost,

$$\sum_{s \leq \tau} (\bar{\beta}^{\tau-t} \bar{v}_{\tau} - \bar{\beta}^{s-t} p_s) x_{is}.$$

Integrating over future possible liquidation dates, at date t , speculator i seeks to maximize expected discounted lifetime trading profits:

$$\max_{\{x_{i\tau}\}_{\tau \geq t}} E_t \left[\sum_{\tau=t}^{\infty} \pi(1-\pi)^{\tau-t} \left[\sum_{s \leq \tau} (\bar{\beta}^{\tau-t} \bar{v}_{\tau} - \bar{\beta}^{s-t} p_s) x_{is} \right] \mid \bar{\mathbf{e}}_{it}, \mathbf{\Omega}_t, \mathbf{x}_{it-1} \right], \quad (1)$$

where we suppress the formal dependence of prices on net order flows. Note that we can decompose speculator i 's expected discounted lifetime trading profits into a (sunk) component due to past trading that he no longer controls and a profit component due to current and future trading that he seeks to maximize. As a result, speculator i 's optimization problem is identical in structure each period.

3 Analysis

Our analysis mirrors that of Back et al. (2000) in that we restrict attention to equilibrium path outcomes. We also focus on stationary linear equilibria. When agents adopt stationary strategies, $\bar{\mathbf{e}}_{it}$ and $\mathbf{\Omega}_t$ fully

determine \mathbf{x}_{it} , so that conditioning on \mathbf{x}_{it-1} is redundant. We first show that we can write a speculator's optimization problem in an analytically tractable way in which the dating on prices, orders and expected contributions to the value process correspond. Intuitively, from the perspective of a speculator at date t , a sufficient statistic for value is the expected liquidation value of the firm given his date- t information.

Lemma 3.1 *Speculator i 's objective can be written as:*

$$\max_{\{x_{i\tau}\}_{\tau \geq t}} E_t \left[\sum_{\tau=t}^{\infty} [\bar{\beta}(1-\pi)]^{\tau-t} \left(\frac{\pi}{1-\rho\bar{\beta}(1-\pi)} \bar{v}_{\tau} - p_{\tau} \right) x_{i\tau} \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t \right].$$

Here, $\frac{\pi}{1-\rho\bar{\beta}(1-\pi)} \bar{v}_{\tau}$ is essentially a one-period security that pays off in period τ , corresponding to the firm's expected liquidation value given date- τ information, integrating over possible liquidation dates. We simplify notation by defining $\beta = \bar{\beta}(1-\pi)$, and working with the adjusted innovations, $e_{i\tau} \equiv \frac{\pi}{1-\rho\bar{\beta}} \bar{e}_{i\tau}$, whose associated variance is σ_e^2 . The asset value is then $v \equiv \frac{\pi}{1-\rho\bar{\beta}} \bar{v}$. This lets us rewrite speculator i 's objective as

$$\max_{\{x_{i\tau}\}_{\tau \geq t}} E_t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left(v_{\tau} - p_{\tau} \right) x_{i\tau} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right]. \quad (2)$$

We next develop the consequences of the stationary linear trading strategies and pricing. We conjecture and verify that the market maker's pricing function is a linear function of the net order flow history:

$$p_t = \sum_{s=0}^{\infty} [\lambda_s(X_{t-s} + u_{t-s})] \equiv \lambda(L)(X_t + u_t).$$

The price function linearly weights current and past net order flow, where λ_s is the weight on date $t-s$ net order flow. To ease presentation, we use the lag operator notation, $\lambda(L)(X_t + u_t)$, to represent this pricing function. We further conjecture that speculators adopt symmetric trading strategies that are linear functions of the histories of their private signals and net order flows, so that trader j 's order at date t takes the form

$$x_{jt} = \sum_{s=0}^{\infty} [b_s e_{jt-s} + B_s(X_{t-s} + u_{t-s})] \equiv b(L)e_{jt} + B(L)(X_t + u_t). \quad (3)$$

That is, b_s is the weight on a signal from date $t-s$, and B_s is the weight on net order flow from date $t-s$. We call b the *private-information trading filter* and call B the *public-information trading filter*.

Given the conjectured linear pricing and linear trading strategies of the other speculators, speculator i computes his optimal order by solving a stationary single-agent optimization problem, which yields his best response. To emphasize the possibility that speculator i *could* choose trading rules other than the conjectured b and B , we index his choices by i ; in equilibrium, $b_i = b$ and $B_i = B$. Using standard time-domain methods, we first prove that trader i 's optimal order is a linear function of the forecast error in the market maker's forecast of his trade on private information. We then exploit the fact that there is a unique solution to the first-order conditions to i 's profit-maximization problem to verify that trading strategies are, in fact, stationary.

Speculator i chooses his order x_{it} at time t to maximize expected discounted trading profits,

$$\max_{x_{it}} E \left[\sum_{s=0}^{\infty} \beta^s \left(v_{t+s} - \lambda(L) \left(\sum_{j=1}^N x_{j,t+s} + u_{t+s} \right) \right) x_{i,t+s} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right].$$

We first solve for the net order flow from the perspective of speculator i to obtain

$$X_t + u_t = x_{it} + \left(\sum_{j \neq i} b(L) e_{jt} + (N-1)B(L)(X_t + u_t) \right) + u_t,$$

where $b(L)$ and $B(L)$ are the conjectured components of the trading strategies of speculators $j \neq i$. Solving for $X_t + u_t$ as a function of speculator i 's order yields

$$\begin{aligned} X_t + u_t &= \frac{1}{1 - (N-1)B(L)} \left(x_{it} + \sum_{j \neq i} b(L) e_{jt} + u_t \right) \\ &\equiv q(L) \left(x_{it} + \sum_{j \neq i} b(L) e_{jt} + u_t \right), \end{aligned}$$

where $q(L) \equiv \frac{1}{1 - (N-1)B(L)}$. The first-order condition for speculator i at time t with respect to x_{it} is

$$0 = E \left[(v_t - \lambda(L)(X_t + u_t)) - \sum_{s=0}^{\infty} \beta^s x_{i,t+s} \sum_{\tau=0}^s \lambda_{\tau} q_{s-\tau} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right],$$

where, for example, q_s denotes the s^{th} lag of $q(L)$. Using

$$q(\beta L^{-1}) \lambda(\beta L^{-1}) x_{it} = \sum_{s=0}^{\infty} \beta^s \sum_{\tau=0}^s \lambda_{\tau} q_{s-\tau} x_{i,t+s},$$

we rewrite the first-order condition as

$$0 = E \left[(v_t - \lambda(L)(X + u_t)) - q(\beta L^{-1}) \lambda(\beta L^{-1}) x_{it} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right].$$

In this first-order condition, beliefs are conditioned on $\boldsymbol{\Omega}_t$, which is endogenous. Proposition 3.2 below details how i 's optimal order depends on $\boldsymbol{\Omega}_t$, establishing that we can circumvent this endogeneity. In particular, it shows that from i 's trades on his private signals, he subtracts off their projections onto the history of prices. The result simply says that speculator i 's relevant private information is the difference between his private signals and what the market maker believes them to be. The proposition rests on a condition—expected discounted profits arbitrarily far into the future converge to zero—that we later prove holds in the equilibrium. The condition amounts to restricting attention to pricing rules that a speculator cannot “game” by buying and selling arbitrarily large orders to earn unbounded profits, something that must hold in any equilibrium.

Proposition 3.2 *Suppose that given date t information, $(\mathbf{e}_{it}, \boldsymbol{\Omega}_t)$, for any $\epsilon > 0$, there exists a future date $t + s(\mathbf{e}_{it}, \boldsymbol{\Omega}_t, \epsilon)$, such that for any $\tau > t + s(\mathbf{e}_{it}, \boldsymbol{\Omega}_t, \epsilon)$, the time t discounted expected profits from date τ on are less than ϵ , i.e., $E_t \left[\sum_{s=\tau}^{\infty} \beta^{s-t} (v_s - p_s) x_{is} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right] < \epsilon$. Then trader i 's order x_{it} is a linear function of the market maker's forecast error (conditional on the net order flow history, $\boldsymbol{\Omega}_t$) of his trade on his private signals, \mathbf{e}_{it} . The first-order condition describing his optimal order is*

$$E \left[q(\beta L^{-1}) \lambda(\beta L^{-1}) x_{it} \middle| \mathbf{e}_{it} - E[\mathbf{e}_{it} | \boldsymbol{\Omega}_t] \right] = E \left[v_t \middle| \mathbf{e}_{it} - E[\mathbf{e}_{it} | \boldsymbol{\Omega}_t] \right]. \quad (4)$$

Using this forecast-error structure, we can drop conditioning on $\boldsymbol{\Omega}_t$ from trader i 's first-order condition. That is, as conjectured, along a stationary equilibrium path, x_{it} is a stationary linear function of \mathbf{e}_{it} —

$E[\mathbf{e}_{it}|\mathbf{\Omega}_t]$, i.e., $B_i(L)$ is a vector of projection coefficients of current and lagged trading intensities on private information onto the net order flow history.

Having proved that equilibrium trading strategies are linear and stationary, we now show that we can re-pose a speculator's conditional profit-maximization problem as an unconditional problem: the two solutions correspond. This lets us analyze the problem in the frequency domain, where the solution is a function reflecting the weighting of current and past information. The analogous first-order condition in the frequency domain takes the form of a functional equation, which is easier to manipulate and characterize.

The unconditional analogue to trader i 's conditional profit-maximization problem given in equation (2) is:

$$\begin{aligned} \max_{b_i, B_i} \quad & E \left[\sum_{t=0}^{\infty} \beta^t \left(V(L) \sum_{j=1}^N e_{jt} - \lambda(L) \left[\left(\sum_{j=1}^N b_j(L) + \sum_{k=1}^N \gamma_k(L) b_j(L) \right) e_{jt} + \sum_{k=1}^N \gamma_k(L) u_t + u_t \right] \right) \right. \\ & \left. \times \left(b_i(L) e_{it} + \gamma_i(L) \left(b_i(L) e_{it} + \sum_{j \neq i} b_j(L) e_{jt} + u_t \right) \right) \right]. \end{aligned} \quad (5)$$

The first line is the market maker's forecast error, $v_t - p_t$, and the second is i 's order, x_{it} . We exploit the linear form of equilibrium trading strategies, posing the optimization as one over the choice of *functions* b_i and B_i (recalling that γ_i is a function of B_i).

Proposition 3.3 below reveals that because equilibrium trading strategies and pricing are stationary and linear—the same linear trading rule (i.e., the coefficients of the linear function) maximizes (2) given *any* equilibrium path of private information and price realizations—the trading strategies that maximize speculator i 's conditional optimization problem, also maximize his unconditional optimization problem. Intuitively, in the conditional problem, speculator i can always choose the unconditional trading rule. But, if i ever finds that unconditional rule suboptimal, then integrating over all possible histories yields an unconditional expected profit that exceeds that attained using the unconditional trading rule. But this contradicts the optimality of the unconditional trading rule. Hence, the conditional and unconditional trading rules must correspond.

Proposition 3.3 *Under the conditions of Proposition 3.2, if b_i and B_i solve speculator i 's unconditional optimization problem, equation (5), then they solve i 's conditional optimization problem, equation (2).*

Rather than solve the unconditional optimization problem directly, it is easier to transform it to the frequency domain where we can use a variational approach. Central to this transformation is the fact that because expectations are unconditional in the objective, we can first integrate out over e_{jt} and u_t to rewrite the objective solely in terms of variances and covariances, plus the strategy functions. To see this, recognize that period profit is the product of the pricing error and the trader's order, each of which we can express as inner products of vectors of weighting functions with the vector of innovations to asset value and liquidity trade. Because e_{jt} and u_t are independently distributed, the expectations of their cross-products are zero at non-identical lags, and yield variance terms at identical lags. Hence, the objective is a sum of cross-products of coefficients of weighting functions and variances. Its structure allows it to be expressed as the

convolution of functions. As a result, we can use the variational methods laid out in the online appendix at <http://econ.uiuc.edu/~bart/frequencydomainmethods> to solve for optimal trading strategies.

The market-maker’s problem. The market maker acts competitively, setting price equal to the expected value of the firm given the history of net order flows. Since period net order flow is normally distributed, conditional forecasts are linear. Again we can obtain the equilibrium pricing function by solving an unconditional optimization problem, i.e., by solving the analogue of (2). That is, the market maker’s pricing function solves the following linear-least-squares prediction problem:

$$\min_{\lambda} E \left[\sum_{t=0}^{\infty} \beta^t (v_t - \lambda(L)(X_t + u_t))^2 \right].$$

Substituting in the form of strategies, we can integrate out the stochastic elements, and solve the market maker’s prediction problem in the frequency domain.

4 Equilibrium

In this section, we first define a stationary, symmetric equilibrium, and then establish its existence.

Definition 4.1 A stationary symmetric linear equilibrium is a triple (b, B, λ) such that:

- (i) b and B solve a trader’s optimization problem.
- (ii) λ solves the competitive market maker’s prediction problem.
- (iii) Net order flow is generated by the optimization by speculators and the market maker.

The solution to the first-order conditions of speculator i ’s optimization problem characterizes his best response to the conjectured actions of the other traders and the price process, mapping the triple (b, B, λ) into best-responses—the functions b_i and B_i . Appendix A presents the variational derivatives characterizing these best responses and that for λ . Lemma 4.2 below shows that both B (and hence γ) and λ are determined by b in equilibrium. Intuitively, we know from Proposition 3.2 that γ is (minus) the projection of a speculator’s trade on his signals onto net order flow (i.e., speculator i ’s net order takes a forecast-error structure). In turn, λ , as the solution to a least squares prediction problem, is the projection of the value process onto net order flow, and because a speculator’s order is determined only by b , net order flow is determined only by b .

Lemma 4.2 γ_i is the projection of b_i onto net order flow, and λ is the projection of the conjectured net order flow onto the value process.

Once we incorporate these equilibrium relationships, equilibrium is determined by the fixed point of speculator i ’s best-response mapping. This mapping is nonlinear, so that closed-form solutions cannot be found directly. This leads us to consider a sequence of iterative best responses by a representative speculator i taking the best response from the k^{th} iteration round as describing the conjectured behavior of the other speculators,

and the pricing relationship consistent with this behavior as the conjectured pricing rule. We then solve for i 's best response to obtain the $(k+1)^{th}$ iteration. We prove that this best-response mapping is a contraction mapping. We then show that the limit of this best-response iteration corresponds to a stationary equilibrium.

Because of the non-linearity of the best-response mapping determining b_i , we establish its fixed point properties indirectly by establishing a fixed point in the informationally-based portion of net order flow. Net order flow consists of two pieces, the direct informationally-based portion, $\sum_{j=1}^N b(L)e_{jt} + u_t$, plus that due to the filtering by speculators of net order flow, $N\gamma(L) \left(\sum_{j=1}^N b(L)e_{jt} + u_t \right)$. It eases presentation to represent the informationally-based portion as a single process,

$$J(L)w_t \equiv \sum_{j=1}^N b(L)e_{jt} + u_t, \quad (6)$$

where w_t is normalized to be a unit-variance i.i.d. fundamental innovation process. Here, w_t is the portion of current net order flow that cannot be forecasted by the market maker from past order flow, and is scaled to have a unit variance. Net order flow is then $(1 + N\gamma(L))J(L)w_t$, and the associated price process is $\lambda(L)(1 + N\gamma(L))J(L)w_t$. To emphasize that the market maker can unwind the filtering of net order flow by speculators, so that his information is effectively $J(L)w_t$, we define

$$\mu \equiv \lambda(L)(1 + N\gamma(L)). \quad (7)$$

The price process is then $\mu(L)J(L)w_t$. This makes clear that, in equilibrium, the market maker filters the informationally-based portion of the order flow process, $J(L)w_t$, directly.

Definition 4.3 Let $H^2(\beta)$ be the space of square-integrable analytic functions on $\Delta(\beta) \equiv \{z : |z| < \beta^{1/2}\}$. Define $T : H^2(\beta) \rightarrow H^2(\beta)$ to be the mapping generated by the equations defining the functions b_i and the corresponding γ , J and μ functions.

Proposition 4.4 presents the contraction-mapping result. The associated unique fixed point characterizes the stationary linear equilibrium.

Proposition 4.4 T has a fixed point b in $H^2(\beta)$. The fixed point b takes the form

$$b = c_0 \left(\frac{c_1}{1 - \rho L} + \sum_{\ell=2}^{\infty} \frac{c_\ell}{1 - a_\ell L} \right)$$

where $c_\ell > 0$ for all ℓ , with $\sum_{\ell=1}^{\infty} c_\ell = 1$ and $a_1 = \rho > a_2 > \dots > a_\ell > \dots > 0$.

The analysis uses a contraction mapping argument on the space of square-integrable functions of a complex variable on the unit circle, which has a tractable structure. Specifically, we develop a contraction argument on iterations of the J function, where $J(L)w_t = \sum_{j=1}^N b(L)e_{jt} + u_t$ is the information-based portion of order flow left after subtracting out the portion based on the filtering of net order flow. The contraction mapping argument assumes that the initial element of the mapping belongs to a restricted subset of H^2 . Specifically,

we begin with a conjectured trading strategy of the form posed above, assuming only that $c_\ell \geq 0$, and that $1 > a_j > 0$ (i.e., allowing trading strategies to have a finite basis, and allowing for $a_1 \neq \rho$). We prove that iterating preserves the structure and that the greatest AR coefficient, i.e., a_1 , converges to ρ . The contraction mapping feature then implies that the fixed point inherits this structure. The contraction mapping implies there is a unique fixed point given the initial class of conjectures, but, because we do not consider arbitrary conjectures, the theorem does not ensure a global uniqueness result. However, Bernhardt and Miao (2004) established that there is a unique linear Markov equilibrium to any finite horizon economy, and we are confident that this uniqueness extends to the stationary economy.

From the fixed point of the contraction mapping, we back out b , γ , and μ . The proposition reveals that private information is used directly in that the first b term matches the autoregressive structure of the value process, V . Intuitively, ρ enters the equilibrium trading strategy because the value process V enters linearly into the first-order condition characterizing a speculator's best response, re-seeding itself, as it were. Subsequent terms in the private-information trading filter feature smaller autoregressive parameters, a_ℓ , implying that speculators trade more aggressively on newer information than old. This reflects that agents' private information conditional on the net order flow history becomes increasingly negatively correlated with age. The fact that all c_ℓ are positive indicates that, in equilibrium, a speculator always trades in the direction of his net private information.

The proof to the proposition shows that the best response to a conjectured trading strategy of the other speculators with k AR(1) terms, is incrementally more sophisticated, with $k + 1$ terms, and the smallest a_ℓ autoregressive coefficient is lower than any of the conjectured ones. It follows that the equilibrium trading strategy does not have a finite representation.⁶ To highlight how it is the strategic one-upmanship by speculators that underlies the complicated nature of equilibrium strategies, note by way of contrast that a monopolist speculator has no incentive to accelerate trading intensity on information. In particular, as a referee pointed out, examining equation (28) in the appendix, substituting $N = 1$ and solving for $J(z)$ yields

$$J(z) = J(0) \frac{1 - f_1 z}{1 - a_1 z}$$

with $a_1 = \rho$ and $f_1 = \frac{\rho}{1 + \sqrt{1 - \beta \rho^2}} < a_1$. Using equation (21) we can then solve for the simple form of a monopolist speculator's trading strategy,

$$b(z) = \frac{\sigma_u}{\sigma_e} \sqrt{1 - \beta \rho^2} \frac{1}{1 - \rho z}.$$

That is, a monopolist speculator's trading intensity mirrors the autoregressive structure of the value process, where the coefficient, $\frac{\sigma_u}{\sigma_e} \sqrt{1 - \beta \rho^2}$, reflects (i) the standard weighting by the noise-to-signal content of net order flow (a result that we generalize to multiple speculators in Proposition 5.1), and (ii) the speculator's intertemporal tradeoff between current and future profits.

⁶Makarov and Rytchkov (2007) consider a variant of our model with a continuum of non-strategic, risk-averse agents. In that setting, they find a related result that the price process does not have a finite representation. In contrast, as Proposition 4.8 reveals, in our strategic setting with a market maker, the price process mirrors the autoregressive structure of the value process.

The next proposition shows that the fixed point identified in Proposition 4.4 characterizes the trading strategies in the stationary linear equilibrium of the trading game. Essentially, the remaining step is to verify that the properties on trading strategies and pricing identified in Proposition 4.4 imply that distant future discounted expected profits can be made arbitrarily small, i.e., the conditions in Proposition 3.2 hold.

Proposition 4.5 *The fixed point of the contraction mapping characterizes the stationary linear equilibrium. In this equilibrium, speculators trade more aggressively on newer information than old, and their trading intensities on private signals decay faster than ρ .*

With the basic properties of the trading filter b established, we now show that the informationally-based portion of net order flow has the same autoregressive structure as b . This result is immediate from its definition, $J(L)w_t = \sum_{j=1}^N b(L)e_{jt} + u_t$, and the fact that liquidity trade is a white noise process. Of course, the coefficients weighting the autoregressive elements for $J(L)$ and $b(L)$ differ, as the informationally-based portion of net order flow is the amalgamation of the contributions of each speculator plus liquidity trade—the market maker has less information than speculators.

Proposition 4.6 *The direct informationally-based order flow process, $J(L)$, has the same autoregressive structure, $(a_1 = \rho, a_2, a_3, \dots)$, as the private trading filter, $b(L)$.*

Proposition 4.7 below details that the functions μ and γ have the same autoregressive structure. Intuitively, this is because both are projections involving the net order flow process— μ is a projection by the market maker of net order flow onto the unobservable value process, while γ is a projection of a speculator’s private information onto the net order flow process. Again, μ and γ have different coefficients weighting the common basis elements, reflecting that speculators have more information than the market maker.

Proposition 4.7 *A trader’s filter $\gamma(L)$ on net order flow has the same autoregressive structure as the market-maker’s filter, $\mu(L)$.*

Finally, we prove that the price process has the same autoregressive structure as the value process. Intuitively, were this not so, systematic deviations between pricing and valuation would arise over time, deviations that speculators would exploit.

Proposition 4.8 *The price process evolves according to*

$$p_t = \rho p_{t-1} + \ell w_t,$$

where $\ell > 0$ is the weighting parameter on the innovation to current net order flow, w_t , i.e., the weighting parameter on the unforecastable portion of the current net order flow.

Formally, this result follows from substituting for the forms of $\mu(L)$ and $J(L)$ into the price process, $\mu(L)J(L)w_t$, and showing that it is proportional to $V(L)w_t$. In essence, the market-maker’s price filter un-

does the complicated autoregressive structure of the order flow induced by speculators' trades and converts it back to a process with the same AR(1) structure as the value process.

5 Trading Intensity Properties

We now characterize the properties of equilibrium trading strategies, pricing and the dynamics of information. We first derive how the levels of noise trade and private information about the asset affect trading strategies, pricing and profits. We prove that the variance of noise trade and the variance of asset value innovations affect equilibrium strategies in a particularly simple way: they scale the trading intensity with which speculators trade on private information.

Fixing all other parameters, let $b(\sigma_e^2, \sigma_u^2)$ be the equilibrium private-information trading intensity function as a function of the variances of asset value innovations and noise trade. Define the equilibrium pricing function, $\lambda(\sigma_e^2, \sigma_u^2)$, and speculator profit function, $\pi(\sigma_e^2, \sigma_u^2)$, analogously. Finally, let $b(1, 1)$ be the equilibrium private-information trading function when $\sigma_e^2 = \sigma_u^2 = 1$. Then,

Proposition 5.1 *Equilibrium private-information trading strategies are linearly homogeneous in $\frac{\sigma_u}{\sigma_e}$:*

$$b(\sigma_e^2, \sigma_u^2) = \frac{\sigma_u}{\sigma_e} b(1, 1) \quad \text{and} \quad \gamma(\sigma_e^2, \sigma_u^2) = \frac{\sigma_u}{\sigma_e} \gamma(1, 1).$$

Proposition 5.1 details that the variances of noise trade and asset innovations do not affect the autoregressive structure of the equilibrium trading strategies. In turn, this means that the impacts of these variances on pricing and informed profits take simple forms:

Proposition 5.2 *The equilibrium pricing function is linearly homogeneous in $\frac{\sigma_e}{\sigma_u}$:*

$$\lambda(\sigma_e^2, \sigma_u^2) = \frac{\sigma_e}{\sigma_u} \lambda(1, 1).$$

The equilibrium profit function is linearly homogeneous in $\sigma_e \sigma_u$:

$$\pi(\sigma_e^2, \sigma_u^2) = \sigma_e \sigma_u \pi(1, 1).$$

Propositions 5.1 and 5.2 reveal that the qualitative impacts of the variances of noise trade and asset innovations found in single-agent settings by Kyle (1985) and Back (1992) are, in fact, general properties.

6 Numerical Characterizations

We now provide quantitative characterizations of equilibrium outcomes, exploring how the extent of competition among speculators interacts with the persistence and correlation in asset innovations to affect (a) how speculators trade on current and past information, (b) speculator profit, (c) pricing, and (d) the information revealed through price about current and past innovations.

Base Case. Our base-case features two speculators with discount factor $\beta = 0.95$. Each e_{jt} is independently and normally distributed with zero mean and variance $\frac{1}{2}$, and the autoregressive parameter of the asset value process is $\rho = 0.97$, so that innovations have a very persistent impact on the asset’s value. Finally, the variance of noise trade each period is one, which matches the total variance of the innovation to the asset’s value.

The market maker’s equilibrium pricing function evolves according to an AR(1) with an autoregressive parameter that matches the value process,

$$\mu(L)J(L)w_t = 0.895 \sum_{\tau=0}^{\infty} (.97)^\tau w_{t-\tau} \leftrightarrow p_t = 0.97p_{t-1} + .895w_t.$$

The leading moving average coefficients of key equilibrium variables are given in Table 1:

Lag	0	1	2	3	4	5	6	7	8	9
$b(L)$	0.355	0.271	0.215	0.177	0.149	0.128	0.112	0.100	0.090	0.081
$\gamma(L)$	-0.101	-0.064	-0.043	-0.030	-0.021	-0.016	-0.012	-0.010	-0.008	-0.007
$\sum_j \pi_{jt-\tau}$	0.359	0.172	0.090	0.051	0.031	0.020	0.013	0.010	0.007	0.005
σ_m^{2FE}	1.095	0.674	0.451	0.323	0.243	0.189	0.152	0.125	0.104	0.088
Cond. Signal Corr.	-0.020	-0.233	-0.425	-0.580	-0.694	-0.775	-0.831	-0.871	-0.898	-0.918

Table 1: Moving average coefficients

Observe that trading intensities on older information quickly decline, especially those on net order flow: By the third lag, the γ coefficients fall by 70%, while the trading intensity on private signals falls by only 50%. This difference reflects that after only a few lags, accounting for his (small) trade on private information in the net order flow ceases to convey much information to i .

Thus, while we prove analytically that trading intensities decline faster than the valuation process—this follows directly from the fact that $\rho = a_1 > a_2 > \dots a_i > \dots 0$ —our numerical analysis quantifies the rapidity with which this occurs. It is not the fact that there is “less” information remaining in older news that drives this decline. Rather, it is that the information being revealed through price results in negative correlations in net private information conditional on the information in price (the last line in the table). Intuitively, at longer lags, price increasingly reflects an average of the private signals, so that when one speculator’s net private information is high, the other’s is low. Hence, they are necessarily negatively correlated. In fact, the correlation in agents’ net private information becomes very negative at short lags; because agents see the current period price, their private information is conditionally negatively correlated even at lag zero. It is this correlation structure that induces speculators to trade more aggressively on recent information than on older information. The effect on trading is an instance of the “waiting game” phenomenon described by Foster and Viswanathan (1996), and Back, Cao and Willard (2000).

It is worthwhile to observe that speculators trade very differently on signals of different vintages than they do in standard models on the same signal at different points in time. These comparisons are subtle, as there are “other” differences between our work and the literature, both in modeling and in numerical implementation.⁷ Nonetheless, the qualitative trading patterns are radically different. With zero correlation in signals and two speculators, Figure 1b, p2135 of Back, Cao and Willard reveals that over the entire first

⁷In our model speculators see contemporaneous price, which causes them to raise trading intensities, as they internalize how their trades affect the information and trades of other speculators. This feature is absent in the discrete time literature [in continuous time, as in Back, Cao and Willard, this timing distinction is irrelevant]. Also, Back, Cao and Willard assume that liquidity trade is persistent, following Brownian motion versus our i.i.d. structure; Brownian motion damps trading intensities.

half of the trading horizon there is a negligible total drop-off in trading intensity on net private information of about 10%; but over the last 5% of the trading horizon trading intensities plummet by roughly 80%, reinforced by the steep increase in price impact, which is going to infinity. In contrast, in our baseline numerical parameterization, trading intensities drop off quickly, but the percentage declines in trading intensities fall at higher lags—exactly the opposite qualitative pattern.

Numerically, we find that only recent information—the first few lags—contributes much to speculator profits. Profits drop off more quickly than do trading intensities both because of the increasingly negative correlation at distant lags, and because private information is revealed over time through trade, so there is “less of it”. This reduction in information is captured by the lag decomposition of the variance of the market maker’s forecast error of the asset’s value. In effect, even though the entire history of signals is available and relevant for the construction of the equilibrium, only recent information is of practical consequence.

This rapid drop-off in profits from older information is very different from what the literature finds when speculators trade over time on their one piece of private information. For example, Foster and Viswanathan (1996) consider 3 speculators with a slight initial correlation in signals of 0.1818 and 800 trading opportunities, and find (see Figure 3, p1462) find that period profit in the first trading opportunity is slightly more than twice that in the 400th trading opportunity—a gradual, almost linear dropoff in profits (equivalently, about 55% of private information is revealed in the first 400 trading opportunities).

Persistence in the value process. Table 2 illustrates how ρ affects equilibrium outcomes in our base-case parameterization. Reducing ρ lowers the contribution of a lagged innovation $e_{jt-\tau}$ to the asset’s period- t value, $\rho^\tau e_{jt-\tau}$, thereby reducing the total private information in the economy, as the variance of the value process is $\frac{N\sigma_e^2}{1-\beta\rho^2}$. Panel 1 illustrates that as ρ is reduced, price becomes far more sensitive to current period order flow. Indeed, even when $\rho = 0.5$, the price impact of order flow quickly approaches what it would be in a static environment ($p_t = X_t + u_t$ when $\rho = 0$). As ρ declines, speculator profits fall, but by far less than proportionately to the reduction in the amount of information. This is because as ρ falls, speculators trade more intensively on newer information (see the lag decompositions of b and γ), and less intensively on older information. As a result, the current innovation contributes more to profits, but all other innovations contribute less. In turn, as ρ is reduced, the market maker’s forecast error falls even more rapidly than do speculator profits: the higher trading intensities on newer information reduce the contribution of current information to the forecast error variance, and lagged innovations matter less.

In essence, when ρ is smaller, future unrevealed private information decays more rapidly, and this shifts the intertemporal tradeoff toward extracting profits at lag 0, and away from deferring profit extraction. The resulting higher trading intensity at lag 0 raises the negative correlation at higher lags to such an extent that by lag 1 reductions in ρ lower trading intensities: by lag 1 the waiting game features dominate the direct effect of the more rapid decline in unrevealed private information in terms of trading intensity incentives. In turn, this implies that reductions in ρ (i) raise lag 0 trading profits but (ii) lead to even sharper declines in trading profits at more distant lags, as there is both less information and reduced speculation at these lags.⁸

Competition and correlation. We next explore how increased competition interacts with the correlation in agent signals to affect equilibrium outcomes. One might conjecture that raising competition by dividing

Finally, to facilitate numerical solution, our base numerical parameterization features a discount factor of $\beta = 0.95$ and slight decay in the value process ($\rho = 0.97$), whereas the literature has $\beta = 1$ and there is no decay in the sole innovation; the discount factor and decay speed up trading in our setting.

⁸We thank a referee for this observation.

Table 2: Persistence and Equilibrium Outcomes

	Aggregate Variables			
	$\rho = 0.97$	$\rho = 0.75$	$\rho = 0.50$	$\rho = 0.00$
Price (p_t)	$.97p_{t-1} + .895(X_t + u_t)$	$.75p_{t-1} + .935(X_t + u_t)$	$.50p_{t-1} + .978(X_t + u_t)$	$(X_t + u_t)$
Informed Profit ($\sum_j \pi_j$)	0.760	0.599	0.542	0.500
Forecast error (σ^{2FE})	3.87	1.017	0.673	0.500
Amount of Information ($\frac{1}{1-\beta\rho^2}$)	9.42	2.14	1.31	1.00

Lag	0	1	2	3	4	5
$b_{je}^{0.97}$	0.355	0.271	0.215	0.177	0.149	0.128
$b_{je}^{0.75}$	0.681	0.390	0.238	0.152	0.100	0.068
$b_{je}^{0.50}$	0.831	0.330	0.141	0.062	0.028	0.012
$\gamma_j^{0.97}$	-0.101	-0.064	-0.043	-0.030	-0.021	-0.016
$\gamma_j^{0.75}$	-0.206	-0.075	-0.030	-0.013	-0.006	-0.003
$\gamma_j^{0.50}$	-0.253	-0.060	-0.014	-0.003	-0.001	-0.000
$\sum_j \pi_j^{0.97}$	0.359	0.172	0.090	0.051	0.031	0.020
$\sum_j \pi_j^{0.75}$	0.490	0.083	0.018	0.004	0.001	0.000
$\sum_j \pi_j^{0.50}$	0.508	0.031	0.002	0.001	0.000	0.000
$\sigma^{2FE0.97}$	1.095	0.674	0.451	0.323	0.243	0.189
$\sigma^{2FE0.75}$	0.731	0.188	0.060	0.021	0.009	0.004
$\sigma^{2FE0.50}$	0.609	0.056	0.006	0.001	0.000	0.000
$\text{corr}^{0.97}$	-0.020	-0.233	-0.425	-0.580	-0.694	-0.775
$\text{corr}^{0.75}$	-0.168	-0.593	-0.829	-0.928	-0.966	-0.981
$\text{corr}^{0.50}$	-0.238	-0.710	-0.919	-0.977	-0.989	-0.992

the total information among more agents should significantly affect aggregate outcomes. Tables 3 and 4 reveal that this conjecture is false unless signals are highly correlated. Indeed, Table 3 reveals that when signals are independently distributed, aggregate profits and the price impact of order flow are extremely insensitive to the degree of competition. One might alternatively conjecture that when signals are independently distributed then the extent of competition should have little effect on strategic interactions—after all, in static settings with independent signals, outcomes do not depend on how information is divided. This conjecture is also wrong—competition greatly affects *how* agents trade on new information relative to old. Finally, one might conjecture that raising competition should lower the market maker’s forecast error variance. This conjecture, too is wrong: increased competition *raises* the forecast error variance if signals are sufficiently uncorrelated.

Table 3 presents equilibrium outcomes for $N = 2, 4$, and 32 agents when e_{jt} is independently distributed with variance $\frac{1}{N}$, so that the variance of total private information is fixed at one. The table reveals that aggregate outcomes—pricing, total speculator profit and market maker forecast error—are remarkably insensitive to how information is divided among agents. Going from two speculators to 32, total speculator profits fall by only two percent; and there is a similar percentage decline in the price impact of order flow. The market maker’s forecast error—a measure of the information revealed through trade—*rises* by five percent due to this increased competition. This is because greater competition causes agents’ private information to be more conditionally negatively correlated. As a result, agents trade less aggressively, so that net order flow contains less information. In sum, in a dynamic setting with independently distributed signals, the extent of competition matters for aggregate outcomes, but not by very much.

The second panel of Table 3 shows how competition affects the lag decomposition of key equilibrium variables when signals are independently distributed. This panel shows that the insensitivity of aggregate outcomes to competition *masks* how agents trade on new information relative to old. Raising competition especially raises the conditional negative correlation in private information at lags, causing agents to tilt trading intensities, raising trading intensities on new information and lowering them on older information. As a result, the source of speculator profits is altered: profits from the current innovation are higher, but future profits are reduced by a little more. So, too, trade reveals more information to the market maker about the current innovation, but less about past innovations, so that forecast error variances at long lags are far higher.

Table 4 shows how correlation interacts with the extent of competition to affect outcomes. When traders’ signals are correlated, the vector version of the frequency-domain methods must be employed (see our online appendix, <http://econ.uiuc.edu/~bart/frequencydomainmethods>.) The vector methods are far more difficult to solve numerically, leading us to set $\rho = 0.5$. Table 4 reveals that correlation has a convex impact on aggregate profits—profits fall increasingly sharply as we increase θ —an effect that is dramatically enhanced by increasing the number of speculators. When there are two traders, raising θ from 0 to 0.5 modestly reduces total expected speculator period profits from 0.533 to 0.506. Thus, neither raising competition alone, nor raising correlation alone has a major impact on aggregate outcomes. Conversely, the number of traders has minimal effects on outcomes only when signal correlations are low: maintaining a high correlation, $\theta = 0.5$, and going from two to four agents sharply reduces total speculator profits by almost 20%. Continuing, with four speculators, a correlation of $\theta = 0.75$ lowers profits 36% below their level when signals are uncorrelated, while with two speculators, profits are reduced by only 14%.

Inspection of the lag decompositions of trading intensities reveals why. With two agents, raising the signal correlation to $\theta = 0.5$ increases trading intensity on current information by 44 percent; but raising both the

Table 3: Competition and Equilibrium Outcomes

	Aggregate Variables		
	N=2	N=4	N=32
Price (p_t)	$0.97p_{t-1} + .895(X_t + u_t)$		
Informed Profit ($\sum_j \pi_j$)	0.760	0.752	0.747
Forecast error (σ^{2FE})	3.87	4.00	4.07

Lag	0	1	2	3	4	5
b_{je}^2	0.355	0.271	0.215	0.177	0.149	0.128
b_{je}^4	0.371	0.267	0.205	0.165	0.137	0.118
b_{je}^{32}	0.382	0.264	0.198	0.158	0.131	0.112
γ_j^2	-0.101	-0.064	-0.043	-0.030	-0.021	-0.016
γ_j^4	-0.051	-0.031	-0.020	-0.014	-0.010	-0.008
γ_j^{32}	-0.006	-0.004	-0.002	-0.002	-0.001	-0.001
$\sum_j \pi_j^2$	0.359	0.172	0.090	0.051	0.031	0.020
$\sum_j \pi_j^4$	0.365	0.166	0.085	0.048	0.029	0.019
$\sum_j \pi_j^{32}$	0.370	0.162	0.082	0.046	0.029	0.019
σ^{2FE2}	1.095	0.674	0.451	0.323	0.243	0.189
σ^{2FE4}	1.073	0.668	0.458	0.335	0.257	0.205
σ^{2FE32}	1.059	0.663	0.460	0.341	0.265	0.212
corr^2	-0.020	-0.233	-0.425	-0.580	-0.694	-0.775
corr^4	-0.049	-0.329	-0.536	-0.677	-0.770	-0.830
corr^{32}	-0.078	-0.400	-0.606	-0.733	-0.810	-0.860

Table 4: Competition and Correlation: $\rho = 0.5$

	$N = 2$ $\theta = 0$	$N=4$ $\theta = 0$	$N=2$ $\theta = \frac{1}{2}$	$N=4$ $\theta = \frac{1}{2}$
Informed Profit ($\sum_j \pi_j$)	0.533	0.532	0.506	0.437

Lag	0	1	2	3	4	5
b_{je}^2 ($\theta = 0$)	.872	.326	.131	.056	.024	.011
b_{je}^4 ($\theta = 0$)	.875	.306	.120	.050	.022	.010
b_{je}^2 ($\theta = .5$)	1.257	.465	.187	.081	.036	.017
b_{je}^4 ($\theta = .5$)	1.768	.538	.203	.085	.038	.018
γ_j^2 ($\theta = 0$)	-.226	-.047	-.011	-.003	-.001	.000
γ_j^4 ($\theta = 0$)	-.113	-.022	-.005	-.001	.000	.000
γ_j^2 ($\theta = .5$)	-.311	-.044	-.008	-.002	-.001	-.000
γ_j^4 ($\theta = .5$)	-.190	-.014	-.002	-.001	-.000	-.000
πj^2 ($\theta = 0$)	.496	.033	.003	.000	.000	.000
πj^4 ($\theta = 0$)	.496	.032	.003	.000	.000	.000
πj^2 ($\theta = .5$)	.488	.018	.001	.000	.000	.000
πj^4 ($\theta = .5$)	.429	.017	.002	.000	.000	.000

number of agents and the signal correlation more than doubles trading intensities on current information. Trading intensities at lags also rise with the increased correlation, but the effect is less. Intuitively, it is the race to trade on common information ahead of other agents that leads to the far higher trading intensities on new information. Trading so aggressively drives down the conditional correlation in agents' information at lags, so that the impact of raising correlation and competition is reduced at lags. It is this more aggressive trading that drives down total speculator profits.

Thus, to significantly affect aggregate outcomes, we must raise both the number of speculators and the correlation in their information. The consequences of doing so are to raise trading intensities and to tilt them toward new information relative to old information, driving down profits.

7 Conclusion

How information is dispersed through prices has long been a central question in finance. We develop a new framework to answer this question. Our model of speculative trade in stock markets resides in an infinite horizon setting, so that our findings are not distorted by finite horizon boundaries. We characterize precisely how information at one date interacts with information from other dates. The use of private information and its revelation through price never ends: each new realization of private information leads agents to re-interpret the history of private and information and prices.

We characterize analytically how the primitives of the economic environment affect equilibrium outcomes, proving that the variances of noise trade and private information proportionately scale trading strategies, pricing, profit and information transmission. We find that competition slows the transmission of information, but the quantitative impacts of competition are slight unless speculator signals are substantially correlated, so that they compete and trade on common information.

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A Proofs and derivations

The structure of this appendix is as follows. First, the proof of Lemma 3.1 shows how to transform a speculator's objective with conventionally-dated assets into one in which the dating of assets, prices, and trades are the same. The proof of Proposition 3.2 then establishes that a speculator's order is equal to the market-maker's forecast error of the speculator's trade on his private signals when projected on price. This lets us to remove the conditioning on public information in a speculator's objective. The proof has two parts. The first is the projection theoretic argument. The second uses a boundedness condition verified in Lemmas A.8-A.9 to preclude the optimality of non-stationary strategies. These lemmas rely on the structure of the functions that characterize the equilibrium: b_i , γ_i , and μ . To establish this structure we derive the first-order conditions that characterize best responses, and analyze the structure dictated by iterating repeatedly on these conditions, starting with a conjecture about the form of trading strategies. Lemmas A.4–A.7 set out some mechanics of frequency-domain analysis, and Lemmas A.1–A.3 establish key properties of the equilibrium functions.

We then prove our central result, Proposition 4.4. This proof establishes that the autoregressive coefficients of $b_i(z)$ are between zero and ρ , the autoregressive coefficient of the asset value process. To do this, we establish that each iteration of the equilibrium functions results in a new set of these coefficients that is interspersed between the previous set. The appendix ends with proofs of our characterization propositions.

A.1 Transforming the objective (Lemma 3.1)

We first decompose the speculator's date- t objective, equation 1, into (i) “sunk” profits from past trading at dates $\tau < t$, plus (ii) profits from current and future trading,

$$\begin{aligned} \max_{\{x_{i\tau}\}_{\tau \geq t}} E_t & \left[\sum_{T=t}^{\infty} \pi(1-\pi)^{T-t} \left[\sum_{\tau \leq T} (\bar{\beta}^{T-t} \bar{v}_T - \bar{\beta}^{\tau-t} p_{\tau}) x_{i\tau} \right] \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t, \mathbf{x}_{it-1} \right] \\ = E_t & \left[\sum_{T=t}^{\infty} \pi(1-\pi)^{T-t} \left[\sum_{\tau < t} (\bar{\beta}^{T-t} \bar{v}_T - \bar{\beta}^{\tau-t} p_{\tau}) x_{i\tau} \right] \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t, \mathbf{x}_{it-1} \right] \\ & + \max_{\{x_{i\tau}\}_{\tau \geq t}} E_t \left[\sum_{T=t}^{\infty} \pi(1-\pi)^{T-t} \left[\sum_{t \leq \tau \leq T} (\bar{\beta}^{T-t} \bar{v}_T - \bar{\beta}^{\tau-t} p_{\tau}) x_{i\tau} \right] \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t, \mathbf{x}_{it-1} \right]. \end{aligned}$$

The speculator maximizes expected lifetime future profits by maximizing expected profits from current and future trading. We expand this component as

$$\begin{aligned} E_t & \left[\pi x_{it} \left[\sum_{T=t}^{\infty} [\bar{\beta}(1-\pi)]^{T-t} \bar{v}_T - \sum_{T=t}^{\infty} (1-\pi)^{T-t} p_t \right] + \pi x_{it+1} \left[\sum_{T=t+1}^{\infty} [\bar{\beta}(1-\pi)]^{T-t} \bar{v}_T - \sum_{T=t+1}^{\infty} (1-\pi)^{T-t} \bar{\beta} p_{t+1} \right] \right. \\ & + \pi x_{it+2} \left[\sum_{T=t+2}^{\infty} [\bar{\beta}(1-\pi)]^{T-t} \bar{v}_T - \sum_{T=t+2}^{\infty} (1-\pi)^{T-t} \bar{\beta}^2 p_{t+2} \right] + \dots \left. \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t \right] \\ = E_t & \left[\pi x_{it} \left[\sum_{T=t}^{\infty} [\bar{\beta}(1-\pi)]^{T-t} \bar{v}_T - \frac{1}{\pi} p_t \right] + \pi x_{it+1} \left[\sum_{T=t+1}^{\infty} [\bar{\beta}(1-\pi)]^{T-t} \bar{v}_T - \frac{\bar{\beta}(1-\pi)}{\pi} p_{t+1} \right] \right. \\ & + \pi x_{it+2} \left[\sum_{T=t+2}^{\infty} [\bar{\beta}(1-\pi)]^{T-t} \bar{v}_T - \frac{[\bar{\beta}(1-\pi)]^2}{\pi} p_{t+2} \right] + \dots \left. \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t \right] \\ = E_t & \left[\sum_{\tau=t}^{\infty} \pi x_{i\tau} \left[\sum_{T=\tau}^{\infty} [\bar{\beta}(1-\pi)]^{T-t} \bar{v}_T - [\bar{\beta}(1-\pi)]^{\tau-t} \frac{p_{\tau}}{\pi} \right] \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t \right]. \end{aligned}$$

The first equality follows from solving the geometric series, and the second from writing the summation compactly. Next, we multiply through by π and factor out the common $\bar{\beta}(1-\pi)^{T-t}$ to rewrite the objective as

$$\begin{aligned} & \max_{\{x_{i\tau}\}_{\tau \geq t}} E_t \left[\sum_{\tau=t}^{\infty} x_{i\tau} [\bar{\beta}(1-\pi)]^{\tau-t} \left(\sum_{T=\tau}^{\infty} [\bar{\beta}(1-\pi)]^{T-\tau} \pi \bar{v}_T - p_{\tau} \right) \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t \right] \\ &= \max_{\{x_{i\tau}\}_{\tau \geq t}} E_t \left[E_{\tau} \left[\sum_{\tau=t}^{\infty} x_{i\tau} [\bar{\beta}(1-\pi)]^{\tau-t} \left(\sum_{T=\tau}^{\infty} [\bar{\beta}(1-\pi)]^{T-\tau} \pi \bar{v}_T - p_{\tau} \right) \middle| \bar{\mathbf{e}}_{i\tau}, \boldsymbol{\Omega}_{\tau} \right] \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t \right]. \end{aligned}$$

The equality follows from iterating on the expectation operator. At date τ , $x_{i\tau}$ is a deterministic function of date τ information (solving i 's optimization problem), so we can pass the date τ expectation operator through:

$$\max_{\{x_{i\tau}\}_{\tau \geq t}} E_t \left[\sum_{\tau=t}^{\infty} x_{i\tau} [\bar{\beta}(1-\pi)]^{\tau-t} E_{\tau} \left[\left(\sum_{T=\tau}^{\infty} [\bar{\beta}(1-\pi)]^{T-\tau} \pi \bar{v}_T - p_{\tau} \right) \middle| \bar{\mathbf{e}}_{i\tau}, \boldsymbol{\Omega}_{\tau} \right] \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t \right].$$

Now use the AR(1) structure of \bar{v}_T , $E_{\tau}[\bar{v}_T | \bar{\mathbf{e}}_{i\tau}, \boldsymbol{\Omega}_{\tau}] = \rho^{T-\tau} E_{\tau}[\bar{v}_{\tau} | \bar{\mathbf{e}}_{i\tau}, \boldsymbol{\Omega}_{\tau}]$, to simplify i 's objective:

$$\begin{aligned} & \max_{\{x_{i\tau}\}_{\tau \geq t}} E_t \left[\sum_{\tau=t}^{\infty} x_{i\tau} [\bar{\beta}(1-\pi)]^{\tau-t} E_{\tau} \left[\left(\sum_{T=\tau}^{\infty} [\rho \bar{\beta}(1-\pi)]^{T-\tau} \pi \bar{v}_{\tau} - p_{\tau} \right) \middle| \bar{\mathbf{e}}_{i\tau}, \boldsymbol{\Omega}_{\tau} \right] \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t \right] \\ &= \max_{\{x_{i\tau}\}_{\tau \geq t}} E_t \left[\sum_{\tau=t}^{\infty} x_{i\tau} [\bar{\beta}(1-\pi)]^{\tau-t} E_{\tau} \left[\left(\frac{\pi}{1 - \rho \bar{\beta}(1-\pi)} \bar{v}_{\tau} - p_{\tau} \right) \middle| \bar{\mathbf{e}}_{i\tau}, \boldsymbol{\Omega}_{\tau} \right] \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t \right]. \end{aligned}$$

The equality follows from solving the geometric series. Finally, we integrate out and rearrange terms to write i 's objective as

$$\max_{\{x_{i\tau}\}_{\tau \geq t}} E_t \left[\sum_{\tau=t}^{\infty} [\bar{\beta}(1-\pi)]^{\tau-t} \left(\frac{\pi}{1 - \rho \bar{\beta}(1-\pi)} \bar{v}_{\tau} - p_{\tau} \right) x_{i\tau} \middle| \bar{\mathbf{e}}_{it}, \boldsymbol{\Omega}_t \right]. \quad \square$$

A.2 Proof of Propositions 3.2 and 3.3

The proof of Proposition 3.2 has two parts. The first uses the theory of recursive projections to simplify a term in the first-order condition, proving that it is orthogonal to public information. The second part uses a boundedness hypothesis that we later verify. We first simplify the $E[v_t | \mathbf{e}_{it}, \boldsymbol{\Omega}_t]$ term in the first-order condition,

$$\begin{aligned} 0 &= E \left[(v_t - \lambda(L)(X + u_t)) - q(\beta L^{-1})\lambda(\beta L^{-1})x_{it} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right] \\ &= E \left[v_t - p_t - q(\beta L^{-1})\lambda(\beta L^{-1})x_{it} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right]. \end{aligned}$$

Using the law of recursive projections, we decompose $E[v_t | \mathbf{e}_{it}, \boldsymbol{\Omega}_t]$ into

$$\begin{aligned} E[v_t | \mathbf{e}_{it}, \boldsymbol{\Omega}_t] &= E[v_t | \boldsymbol{\Omega}_t] + E \left[v_t - E[v_t | \boldsymbol{\Omega}_t] \middle| \mathbf{e}_{it} - E[\mathbf{e}_{it} | \boldsymbol{\Omega}_t] \right] \\ &= p_t + E \left[v_t - E[v_t | \boldsymbol{\Omega}_t] \middle| \mathbf{e}_{it} - E[\mathbf{e}_{it} | \boldsymbol{\Omega}_t] \right], \end{aligned}$$

using $p_t = E[v_t | \boldsymbol{\Omega}_t]$. By construction, $\boldsymbol{\Omega}_t$ is orthogonal to the forecast error $\mathbf{e}_{it} - E[\mathbf{e}_{it} | \boldsymbol{\Omega}_t]$, so that

$$E \left[E[v_t | \boldsymbol{\Omega}_t] \middle| \mathbf{e}_{it} - E[\mathbf{e}_{it} | \boldsymbol{\Omega}_t] \right] = 0,$$

and hence

$$E[v_t | \mathbf{e}_{it}, \boldsymbol{\Omega}_t] = p_t + E \left[v_t \middle| \mathbf{e}_{it} - E[\mathbf{e}_{it} | \boldsymbol{\Omega}_t] \right].$$

Substituting into the first-order condition, the p_t terms cancel, so that the sum of the first two terms in the first-order condition simplifies to $E \left[v_t \middle| \mathbf{e}_{it} - E \left[\mathbf{e}_{it} \middle| \boldsymbol{\Omega}_t \right] \right]$, which is orthogonal to $\boldsymbol{\Omega}_t$. Thus, we have

$$E \left[q(\beta L^{-1}) \lambda(\beta L^{-1}) x_{it} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right] = \left[v_t \middle| \mathbf{e}_{it} - E \left[\mathbf{e}_{it} \middle| \boldsymbol{\Omega}_t \right] \right].$$

Since the right-hand side only varies with e_{it} and Ω according to the structure of forecast errors, $\mathbf{e}_{it} - E \left[\mathbf{e}_{it} \middle| \boldsymbol{\Omega}_t \right]$, so must the forecast of future trades on the left-hand side, i.e., the first-order condition takes the form given in the Proposition.

It remains to prove that the optimal orders $x_{it+s}, s > 0$, are characterized by first-order conditions. Suppose to the contrary that they are not. Then they must generate discounted time- t expected profits that exceed by some $\epsilon > 0$ the maximized expected profits given the restriction that orders not depend on the net order flow history. We decompose i 's discounted expected time- t profits into

$$E \left[\sum_{s=0}^{\tau-1} \beta^s (v_{t+s} - p_{t+s}) x_{it+s} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right] + E \left[\sum_{s=\tau}^{\infty} \beta^s (v_{t+s} - p_{t+s}) x_{it+s} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right]. \quad (8)$$

Using the boundedness premise, as a function of date t information, for all $\tau \geq t+s(\mathbf{e}_{it}, \boldsymbol{\Omega}_t, \epsilon)$, the discounted time- t expected profits from the second sum are strictly less than ϵ . Hence, to derive a contradiction, we need only show that the strategy maximizing the first sum in (8), $E \left[\sum_{s=0}^{\tau-1} \beta^s (v_{t+s} - p_{t+s}) x_{it+s} \middle| \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right]$, is solely a function of the forecast error, $\mathbf{e}_{it} - E \left[\mathbf{e}_{it} \middle| \boldsymbol{\Omega}_t \right]$. But, our setting is a special case of Bernhardt and Miao (2004), who establish that optimal/equilibrium trading strategies are linear functions of forecast errors in a general finite horizon setting, with arbitrary numbers of informed speculators at each date, and arbitrary correlations in the signals that speculators receive at each date. \square

Proof: (Proposition 3.3.) Let (b_i^c, B_i^c) be the stationary trading rules that solve the conditional optimization problem (2), and suppose that they did not correspond to the trading rules (b_i^u, B_i^u) that maximize i 's unconditional optimization problem (5). Then for a set of histories of positive probability, (b_i^c, B_i^c) must earn higher profits than (b_i^u, B_i^u) . Moreover, there are no histories for which the reverse is true: the trading rules that maximize profits given any history $(\mathbf{e}_{jt}, \boldsymbol{\Omega}_t)$ must yield expected profits that are at least as high as those from using (b_i^u, B_i^u) , because (b_i^u, B_i^u) can feasibly be employed at any history.

Integrating over all possible histories yields i 's unconditional payoff from using the rule (b_i^c, B_i^c) . But, this then implies that the unconditional payoff from using (b_i^c, B_i^c) exceeds that from using (b_i^u, B_i^u) . This contradicts the premise that (b_i^u, B_i^u) maximize unconditional expected profits. \square

A.3 Frequency-Domain Analysis

We now prove that an equilibrium exists and characterize it. We write speculator i 's optimization problem in the frequency domain as

$$\max_{b_i(\cdot), B_i(\cdot)} \frac{1}{2\pi i} \oint \text{tr} \left\{ \begin{pmatrix} V(z) - \lambda(1 + (N-1)\gamma(z) + \gamma_i(z))b(z) \\ \vdots \\ V(z) - \lambda(1 + (N-1)\gamma(z) + \gamma_i(z))b_i(z) \\ \vdots \\ V(z) - \lambda(1 + (N-1)\gamma(z) + \gamma_i(z))b(z) \\ -\lambda(z)(1 + (N-1)\gamma(z) + \gamma_i(z)) \end{pmatrix} \right. \\ \left. \times \begin{pmatrix} \gamma_i(\beta z^{-1})b(\beta z^{-1}) & \dots & (1 + \gamma_i(\beta z^{-1}))b_i(\beta z^{-1}) & \dots & \gamma_i(\beta z^{-1})b(\beta z^{-1}) & \gamma_i(\beta z^{-1}) \end{pmatrix} \begin{pmatrix} \sigma_e^2 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & \sigma_e^2 & 0 \\ 0 & 0 & \dots & 0 & \sigma_u^2 \end{pmatrix} \right\} \frac{dz}{z},$$

where the integration is counterclockwise around the unit circle. We abbreviate notation in what follows, for example writing b_i instead of $b_i(z)$, and b_i^* instead of $b_i(\beta z^{-1})$. The objective takes a vector form because there are $N + 1$ fundamental processes: the N innovation processes $\{e_{jt}\}$, and the noise trade process u_t . The final term in the objective reflects that we have passed the expectations operator through: it is the variance covariance matrix of signals and noise trade. The column vector is the market maker's forecast error of the firm's value, and the row vector multiplying it is i 's order.

The market-maker's frequency-domain objective is the following linear, least-squares prediction problem,

$$\min_{\lambda(\cdot)} \frac{1}{2\pi i} \oint \text{tr} \left\{ \begin{pmatrix} V - \lambda b(1 + N\gamma) \\ \vdots \\ V - \lambda b(1 + N\gamma) \\ -\lambda(1 + N\gamma) \end{pmatrix} \right. \\ \left. \times \begin{pmatrix} (V^* - \lambda^* b^*(1 + N\gamma^*)) & \dots & V^* - \lambda^* b^*(1 + N\gamma^*) & -\lambda^*(1 + N\gamma^*) \end{pmatrix} \begin{pmatrix} \sigma_e^2 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & \sigma_e^2 & 0 \\ 0 & 0 & \dots & 0 & \sigma_u^2 \end{pmatrix} \right\} \frac{dz}{z},$$

It eases presentation to solve the market maker's optimization problem first. To do so, we construct a variation $\lambda_i + \alpha\zeta$. Taking the variational derivative for λ and exploiting β -symmetry (see our online appendix, <http://econ.uiuc.edu/~bart/frequencydomainmethods>) we obtain the following Wiener-Hopf equation as the first-order condition for λ :

$$N(V - \lambda b(1 + N\gamma))b^*(1 + N\gamma^*)\sigma_e^2 - \lambda(1 + N\gamma)(1 + N\gamma^*)\sigma_u^2 = \sum_{-\infty}^{-1}. \quad (9)$$

We follow standard convention and use $\sum_{-\infty}^{-1}$ as shorthand for an arbitrary function that has only negative powers of z in its Laurent expansion, and hence cannot be part of the solution to an agent's optimization

problem. Because the covariance matrix is diagonal, the first-order condition takes a one-dimensional form. Dividing (9) by $(1 + N\gamma^*)$ and rearranging yields

$$N(Vb^* - \lambda(1 + N\gamma)bb^*)\sigma_e^2 - \lambda(1 + N\gamma)\sigma_u^2 = \sum_{-\infty}^{-1}. \quad (10)$$

Solution for b_i . Taking the variational derivative of speculator i 's objective with respect to b_i and exploiting β -symmetry (see our online appendix, <http://econ.uiuc.edu/~bart/frequencydomainmethods>) yields the following Wiener-Hopf equation:

$$[(V - \lambda b_i(1 + (N - 1)\gamma + \gamma_i)(1 + \gamma_i^*) - \lambda^* b_i(1 + (N - 1)\gamma^* + \gamma_i^*)(1 + \gamma_i))]\sigma_e^2 = \sum_{-\infty}^{-1}. \quad (11)$$

Substituting for the posited symmetric equilibrium trading strategies, $b_i = b$ and $\gamma_i = \gamma$, equation (11) simplifies to

$$[(V - \lambda b(1 + N\gamma)(1 + \gamma^*) - \lambda^* b(1 + N\gamma^*)(1 + \gamma))]\sigma_e^2 = \sum_{-\infty}^{-1}. \quad (12)$$

Re-arranging equation (12), yields

$$b[\lambda(1 + N\gamma)(1 + \gamma) + \lambda^*(1 + N\gamma^*)(1 + \gamma)]\sigma_e^2 = V(1 + \gamma^*)\sigma_e^2 + \sum_{-\infty}^{-1}.$$

The coefficient of σ_e^2 on the left-hand side is the sum of complex conjugates. Due to this symmetry, by Rozanov's theorem (1967), we can factor it into the product of an analytic, invertible function $g(z)$ and its conjugate,⁹

$$gg^* \equiv \lambda(1 + N\gamma)(1 + \gamma^*) + \lambda^*(1 + N\gamma^*)(1 + \gamma). \quad (13)$$

Substituting for gg^* , (12) becomes

$$b(gg^*) = V(1 + \gamma^*) + \sum_{-\infty}^{-1} \Rightarrow b = g^{-1}[g^{*-1}V(1 + \gamma^*)]_+. \quad (14)$$

Here $[\cdot]_+$ is the *annihilator operator* that sets the coefficients of negative powers of z in the Laurent expansion to zero, while preserving all coefficients on the non-negative powers of z to obtain a feasible (backward-looking in time) solution to the speculator's optimization problem.

Solution for B_i . We first recall the definition (translated to the frequency domain) of γ_i :

$$\gamma_i = \frac{B_i(z)}{1 - \sum_{j \neq i} B_j(z) - B_i(z)}. \quad (15)$$

We note the following Frechet derivatives of γ_j and $1 + N\gamma$ with respect to B_i :

$$\frac{\partial \gamma_i}{\partial B_i} = \frac{1}{1 - \sum_k B_k} + \frac{B_i}{(1 - \sum_k B_k)^2} = (1 + \gamma_i)(1 + N\gamma) \quad \text{and} \quad \frac{\partial(1 + N\gamma)}{\partial B_i} = (1 + N\gamma)^2. \quad (16)$$

⁹We remind the reader that the conjugate of $g(z)$ is $g(\beta z^{-1})$. Technically, this is not the complex conjugate per se, but can effectively be treated as such; see the online appendix <http://econ.uiuc.edu/~bart/frequencydomainmethods>.

Taking the variational derivative with respect to B_i yields

$$\begin{aligned}
& [(Vb^* - \lambda b(1 + (N-1)\gamma + \gamma_i))b^*(1 + \gamma_i^*)(1 + N\gamma^*) - \lambda^*b^*b\gamma_i(1 + N\gamma^*)^2]\sigma_e^2 + \dots \\
& + [(Vb^* - \lambda b(1 + (N-1)\gamma + \gamma_i))b^*(1 + \gamma_i^*)(1 + N\gamma^*) - \lambda^*b^*b(1 + \gamma_i)(1 + N\gamma^*)^2]\sigma_e^2 + \dots \\
& + [(Vb^* - \lambda b(1 + (N-1)\gamma + \gamma_i))b^*(1 + \gamma_i^*)(1 + N\gamma^*) - \lambda^*b^*b\gamma_i(1 + N\gamma^*)^2]\sigma_e^2 \\
& + [-\lambda(1 + (N-1)\gamma + \gamma_i)(1 + \gamma_i^*)(1 + N\gamma^*) - \lambda^*\gamma_i(1 + N\gamma^*)^2]\sigma_u^2 = \sum_{-\infty}^{-1}. \quad (17)
\end{aligned}$$

Substituting for $b_i = b$ and $\gamma_i = \gamma$ this first-order condition simplifies to

$$\begin{aligned}
& [N[(Vb^* - \lambda b(1 + N\gamma)b^*)(1 + \gamma^*)(1 + N\gamma^*) - \lambda^*b^*b\gamma(1 + N\gamma^*)^2] - \lambda^*b^*b(1 + N\gamma^*)^2]\sigma_e^2 \\
& + [-\lambda(1 + N\gamma)(1 + \gamma^*)(1 + N\gamma^*) - \lambda^*\gamma(1 + N\gamma^*)^2]\sigma_u^2 = \sum_{-\infty}^{-1}.
\end{aligned}$$

Next, divide out a factor $(1 + N\gamma^*)$ to obtain

$$\begin{aligned}
& [N[(Vb^* - \lambda b(1 + N\gamma)b^*)(1 + \gamma^*) - \lambda^*b^*b\gamma(1 + N\gamma^*)] - \lambda^*b^*b(1 + N\gamma^*)]\sigma_e^2 \\
& + [-\lambda(1 + N\gamma)(1 + \gamma^*) - \lambda^*\gamma(1 + N\gamma^*)]\sigma_u^2 = \sum_{-\infty}^{-1}. \quad (18)
\end{aligned}$$

Re-arranging yields

$$\begin{aligned}
& [N(Vb^* - \lambda(1 + N\gamma)bb^*)\sigma_e^2 - \lambda(1 + N\gamma\sigma_u^2)](1 + \gamma^*) \\
& - \lambda^*Nb^*b\gamma(1 + N\gamma^*)\sigma_e^2 - \lambda^*b^*b(1 + N\gamma^*)\sigma_e^2 - \lambda^*\gamma(1 + N\gamma^*)\sigma_u^2 = \sum_{-\infty}^{-1}. \quad (19)
\end{aligned}$$

The first part of this expression, the coefficient of $(1 + \gamma^*)$, duplicates the market-maker's first-order condition (10) and drops out. We can divide λ^* and $(1 + N\gamma^*)$ out of the remaining term, yielding

$$-Nb^*b\gamma\sigma_e^2 - b^*b\sigma_e^2 - \gamma\sigma_u^2 = \sum_{-\infty}^{-1}. \quad (20)$$

To solve this, we need a factorization step. We first write the direct informationally-based portion of net order flow as $J(L)w_t = \sum_{j=1}^N b(L)e_{jt} + u_t$, where w_t is normalized to be a unit-variance i.i.d. innovation process. We then have

$$JJ^* = Nbb^*\sigma_e^2 + \sigma_u^2, \quad (21)$$

where by Rozanov's factorization theorem, we can choose $J(z)$ to be analytic and invertible. Hence, the inverse, $J(z)^{-1}$, is analytic. Substituting (21) into (20) yields

$$J^*J\gamma = -bb^*\sigma_e^2. \quad (22)$$

We have thus demonstrated the following:

Lemma A.1 *Speculator i 's trade on net order flow takes the form of a projection: $\gamma = -J^{-1}[J^{*-1}bb^*\sigma_e^2]_+$.*

This lemma re-establishes our finding in the time domain that γ_i is the projection of i 's trades on private information onto the net order flow process. The negative sign in the formula for γ_i emphasizes that i subtracts this projected information from his gross order flow— i trades less aggressively to the degree that his private information can be inferred from the net order flow history.

Finally, substituting for JJ^* into the first-order condition for λ , equation (10) yields

$$\lambda JJ^*(1 + N\gamma) = V(Nb^*\sigma_e^2) + \sum_{-\infty}^{-1}. \quad (23)$$

The price process is then $\lambda(L)(1 + N\gamma(L))J(L)w_t$. Defining $\mu \equiv \lambda(1 + N\gamma)$, we solve (23) for

$$\mu \equiv \lambda(1 + N\gamma) = J^{-1}[J^{*-1}V(Nb^*\sigma_e^2)]_+. \quad (24)$$

The following lemma shows that the annihilator operator can generate explicit solutions if the argument of the operator has a special form.

Lemma A.2 *Let f be analytic on $\beta^{-1/2}$ and $\rho < \beta^{-1/2}$. Then $[f^*(1 - \rho z)^{-1}]_+ = f(\beta\rho)(1 - \rho z)^{-1}$.*

Proof: Direct computation (see Taub (1986)). \square

A.3.1 Properties of the solution.

Lemma A.3 $1 + N\gamma$ is invertible.

Proof: Solve the definition of J in (21) for $bb^*\sigma_e^2 = \frac{JJ^* - \sigma_u^2}{N}$ and substitute this into the solution for γ given in Lemma A.1, to obtain

$$\begin{aligned} N\gamma &= -J^{-1}[J^{*-1}(JJ^* - \sigma_u^2)]_+ \\ &= -1 + J^{-1}[J^{*-1}\sigma_u^2]_+ \Rightarrow 1 + N\gamma = J^{-1}[J^{*-1}\sigma_u^2]_+. \end{aligned}$$

Because $[J^{*-1}\sigma_u^2]_+$ is a scalar and J^{-1} is invertible by construction, the result follows. \square

Exploiting Lemma A.3, we substitute J into the first-order condition for λ and invert to obtain

$$\lambda = (1 + N\gamma)^{-1}J^{-1}[J^{*-1}VNb^*\sigma_e^2]_+. \quad (25)$$

It follows that λ is invertible: $(1 + N\gamma)^{-1}$ and J^{-1} are invertible by construction, and because V is an AR (1) process, $[J^{*-1}VNb^*\sigma_e^2]_+$ is proportional to V (Lemma A.2), which is invertible. This invertibility implies that observing prices is equivalent to observing the J process.

A.3.2 Some lemmas about rational functions

Lemmas A.4-A.7, establish properties of rational functions that we need for the contraction mapping argument used to prove Proposition 4.4. The contraction argument requires us to bound appropriately some structured polynomials. To do this, we first characterize the behavior of the roots of symmetric quadratic equations that arise in factorizations.

Lemma A.4 *Let $0 < a < 1$. Define $f(z)$ by $f(z)f(z^{-1}) \equiv (1 - az)(1 - az^{-1}) + \sigma^2$. Then $f(z) = f_0(1 - f_1z)$ and $f_1 < a$.*

Proof: Multiplying out and matching coefficients of z^0 and $z + z^{-1}$ respectively yields the equations

$$\sigma^2 + 1 + a^2 = f_0^2(1 + f_1^2) \quad \text{and} \quad f_0^2 f_1 = a,$$

which can be solved for f_0 and f_1 . The solution for f_1 is

$$f_1 = \frac{\frac{\sigma^2+1+a^2}{a} \pm \sqrt{\left(\frac{\sigma^2+1+a^2}{a}\right)^2 - 4}}{2}.$$

Because the root must be fractional, the relevant root is the smaller one, i.e., the radical is subtracted. Routine algebra then reveals that

$$\frac{\frac{\sigma^2+1+a^2}{a} - \sqrt{\left(\frac{\sigma^2+1+a^2}{a}\right)^2 - 4}}{2} < a. \quad \square$$

Lemma A.4 shows that adding an MA process to an independent i.i.d. process results in a moving-average component of the joint process that has a smaller MA parameter than the initial MA parameter. The next lemma establishes a similar result: adding an AR process to an independent i.i.d. process results in a moving-average component of the joint process that has a smaller MA parameter than the initial AR parameter. We use this result to establish the characteristics of $J(z)$.

Lemma A.5 *Let $h(z) = \prod_{i=1}^{k-1}(1 - f_i z) / \prod_{i=1}^k(1 - a_i z)$ with $1 > a_1 > f_1 > a_2 > f_2 > \dots > a_k > 0$. The partial-fractions representation of $h(z)$ is*

$$h(z) = \sum_{i=1}^k \frac{c_i}{1 - a_i z}, \quad \text{where} \quad \sum_{i=1}^k c_i = 1 \quad \text{and} \quad c_i > 0.$$

Conversely, this partial-fractions representation implies the factored-form representation with $1 > a_1 > f_1 > a_2 > f_2 > \dots > a_k > 0$.

Proof: We prove the result by recursively expanding $\prod_{i=1}^{k-1}(1 - f_i z) / \prod_{i=1}^k(1 - a_i z)$ into partial-fractions form. We first expand $\frac{1-f_1 z}{(1-a_1 z)(1-a_2 z)}$ into $\frac{c_1^1}{1-a_1 z} + \frac{c_2^1}{1-a_2 z}$, where

$$(i) \quad c_1^1 = \frac{a_1 - f_1}{a_1 - a_2} \in (0, 1); \quad (ii) \quad c_2^1 = \frac{f_1 - a_2}{a_1 - a_2} \in (0, 1); \quad \text{and} \quad (iii) \quad c_1^1 + c_2^1 = 1,$$

proving the result for $k = 1$. We now assume that the induction hypothesis holds for k , i.e., there is a partial-fractions representation with the appropriate properties on c_i^k , and prove that it holds for $k + 1$. We have

$$\frac{\prod_{i=1}^k(1 - f_i z)}{\prod_{i=1}^{k+1}(1 - a_i z)} = \frac{\prod_{i=1}^{k-1}(1 - f_i z)}{\prod_{i=1}^k(1 - a_i z)} \frac{(1 - f_k z)}{(1 - a_{k+1} z)}.$$

Substituting the posited partial-fractions representation for k yields

$$\sum_{i=1}^k \frac{c_i^k(1 - f_k z)}{(1 - a_i z)(1 - a_{k+1} z)} = \sum_{i=1}^k c_i^k \left[\frac{c_{i1}^{k+1}}{(1 - a_i z)} + \frac{c_{i2}^{k+1}}{(1 - a_{k+1} z)} \right],$$

where $c_{i1}^{k+1} + c_{i2}^{k+1} = 1$ and $c_{ij}^{k+1} < 1$, and because, by induction, $\sum_{i=1}^k c_i^k = 1$, then

$$\sum_{i=1}^k c_i^k (c_{i1}^{k+1} + c_{i2}^{k+1}) \equiv \sum_{i=1}^{k+1} c_i^{k+1} = 1.$$

To establish the converse argument, first assume a partial-fractions representation for $k = 2$,

$$\frac{c_1}{1 - a_1 z} + \frac{c_2}{1 - a_2 z},$$

with $c_1 + c_2 = 1$ and $c_1, c_2 > 0$, and $a_1 > a_2$. Establishing a common denominator yields

$$\frac{c_1}{1 - a_1 z} + \frac{c_2}{1 - a_2 z} = \frac{1 - (c_1 a_2 + c_2 a_1)z}{(1 - a_1 z)(1 - a_2 z)},$$

so that $a_1 > c_2 a_1 + c_1 a_2 = f_1 > a_2$. An analogous induction argument establishes this representation holds for an arbitrary k . \square

Lemma A.6 *Let $-\rho < f < \rho$ and $0 < \rho < 1$. Then*

$$\inf_{\{|z|=1\}} 2 \operatorname{Re} \left(\frac{1 - fz}{1 - f\rho} \right) > 1.$$

Proof: Using the polar form $z = e^{i\theta} = \cos(\theta) + i \sin(\theta)$, if $0 \leq f < \rho$, then

$$\inf_{\{|z|=1\}} \operatorname{Re}(1 - fz) = \inf_{\{|z|=1\}} (1 - f \cos(\theta)), \quad (26)$$

with the infimum clearly attained at $\theta = 0$, i.e., at $z = 1$. Hence,

$$\inf_{\{|z|=1\}} 2 \operatorname{Re} \left(\frac{1 - fz}{1 - f\rho} \right) \geq \frac{2(1 - f)}{1 - f\rho} > 1,$$

for $f \in (0, \rho)$. Similarly, if $-\rho < f < 0$, the infimum is attained at $z = -1$, and we again have

$$\inf_{\{|z|=1\}} 2 \operatorname{Re} \left(\frac{1 - fz}{1 - f\rho} \right) \geq \frac{2(1 + f)}{1 - f\rho} > 1. \quad \square$$

Lemma A.7 *Let $0 < f_1 < a < f_2 < \rho$. Then*

$$\inf_{\{|z|=1\}} 2 \operatorname{Re} \left(\frac{\frac{(1 - f_1 z)(1 - f_2 z)}{1 - az}}{\frac{(1 - f_1 \rho)(1 - f_2 \rho)}{1 - a\rho}} \right) > 1.$$

Proof: We first expand the denominator of $\operatorname{Re} \left(\frac{(1 - f_1 z)(1 - f_2 z)}{1 - az} \right)$ in a power series,

$$\begin{aligned} \operatorname{Re} \left(\frac{(1 - f_1 z)(1 - f_2 z)}{1 - az} \right) &= \operatorname{Re} \left((1 - (f_1 + f_2)z + f_1 f_2 z^2) \sum_{k=0}^{\infty} a^k z^k \right) \\ &= \operatorname{Re} \left(\sum_{k=0}^{\infty} a^k z^k - (f_1 + f_2) \sum_{k=1}^{\infty} a^{k-1} z^k + f_1 f_2 \sum_{k=2}^{\infty} a^{k-2} z^k \right) \\ &= \left(1 - \frac{f_1 + f_2}{a} + \frac{f_1 f_2}{a^2} \right) \operatorname{Re} \left(\sum_{k=0}^{\infty} a^k z^k \right) + \operatorname{Re} \left(\frac{f_1 + f_2}{a} - \frac{f_1 f_2}{a^2} - \frac{f_1 f_2}{a} z \right) \\ &= \left(1 - \frac{f_1 + f_2}{a} + \frac{f_1 f_2}{a^2} \right) \operatorname{Re} \left(\frac{1}{1 - az} \right) + \left(\frac{f_1 + f_2}{a} - \frac{f_1 f_2}{a^2} - \frac{f_1 f_2}{a} \operatorname{Re}(z) \right). \end{aligned}$$

We now evaluate the first term. Observe that for any complex number $a + ib$, the real part of the inverse is the sum of the complex conjugates:

$$\operatorname{Re} \left(\frac{1}{a + bi} \right) = \frac{1}{a + bi} + \frac{1}{a - bi} = \frac{2a}{a^2 + b^2}.$$

Using $1 - az = 1 - a \cos(\theta) - ia \sin(\theta)$, we have

$$\begin{aligned} \operatorname{Re} \left(\frac{1}{1 - az} \right) &= \frac{2(1 - a \cos(\theta))}{(1 - a \cos(\theta))^2 + (a \sin(\theta))^2} = \frac{2(1 - a \cos(\theta))}{1 - 2a \cos(\theta) + a^2 \cos^2(\theta) + a^2 \sin^2(\theta)} \\ &= \frac{2(1 - a \cos(\theta))}{1 - 2a \cos(\theta) + a^2}, \end{aligned}$$

Substituting this structure, and using $\operatorname{Re}(z) = \cos(\theta)$ we obtain

$$\operatorname{Re} \left(\frac{(1 - f_1 z)(1 - f_2 z)}{1 - az} \right) = \left(1 - \frac{f_1 + f_2}{a} + \frac{f_1 f_2}{a^2} \right) \frac{2(1 - a \cos(\theta))}{1 - 2a \cos(\theta) + a^2} + \left(\frac{f_1 + f_2}{a} - \frac{f_1 f_2}{a^2} - \frac{f_1 f_2}{a} \cos(\theta) \right).$$

We now show that $\operatorname{Re} \left(\frac{(1 - f_1 z)(1 - f_2 z)}{1 - az} \right)$ achieves its maximum at $\theta = 0$. Differentiating the last term with respect to θ yields

$$\frac{d}{d\theta} \left(-\frac{f_1 f_2}{a} \cos(\theta) \right) = \frac{f_1 f_2}{a} \sin(\theta) > 0.$$

The other term is slightly more complicated. We have

$$\frac{d}{dx} \frac{2(1 - ax)}{1 - 2ax + a^2} = \frac{a(1 - a^2)}{(1 - 2ax + a^2)^2} > 0,$$

and because $\frac{d}{d\theta} x = \frac{d}{d\theta} \cos(\theta) = -\sin(\theta) < 0$, the derivative of the first term is positive if the coefficient

$$\left(1 - \frac{f_1 + f_2}{a} + \frac{f_1 f_2}{a^2} \right) \equiv \frac{1}{a^2} (a^2 - a(f_1 + f_2) + f_1 f_2) \equiv \frac{1}{a^2} X(a)$$

is negative. We know that $f_1 < a < f_2$, so we just need to show that $X(a) < 0$. It is immediate that $X(f_1) = X(f_2) = 0$. Hence $X(a) < 0$, if $X''(a) > 0$. But $X''(a) = 2 > 0$, so this follows.

Finally, substitute in the fact that the infimum is attained at $\theta = 0$ (or equivalently $z = 1$) to bound

$$\inf_{\{|z|=1\}} 2 \operatorname{Re} \left(\frac{(1 - f_1 z)(1 - f_2 z)}{1 - az} \right) = \inf_{\{|z|=1\}} 2 \operatorname{Re} \left(\frac{\frac{1 - f_1 z}{1 - f_1 \rho} \frac{1 - f_2 z}{1 - f_2 \rho}}{\frac{1 - az}{1 - a\rho}} \right) = \frac{1 - f_1}{1 - f_1 \rho} \frac{2(1 - f_2)}{1 - f_2 \rho}$$

from below. From Lemma A.6, the second term, $\frac{2(1 - f_2)}{1 - f_2 \rho}$ exceeds one. To see that first term exceeds one note that $\frac{1 - y}{1 - y\rho}$ falls in y : $0 < f_1 < a$ implies that the fraction in the numerator exceeds that in the denominator. \square

Lemma A.7 extends to higher-order ratios—applying a partial-fractions expansion inductively reveals that the infimum is always attained at $z = 1$, and the same ratio argument can be repeatedly applied.

A.4 Proof of Proposition 4.4

We now prove our main result, Proposition 4.4. We first develop a fixed point condition for the function JJ^* using the equations for b , J , μ , γ , and g (equations (14), (21), (23), Lemma A.1, and (13), respectively). We then prove that if J has a structure such that its numerator and denominator coefficients are real and initially have an interspersing structure, then this structure is preserved by the mapping implicit in the fixed point condition; moreover, with this structure, the modulus of the mapping is a positive fraction, establishing the contraction property. The proof has several sub-arguments that we develop in sequence.

The Recursion. Making use of the AR structure of the value process filter V , we first apply the annihilator lemma, Lemma A.2, to the solution for b in (14) to obtain

$$b = g^{-1} [g^{*-1} (1 + \gamma^*) V]_+ = g^{-1} A_b V,$$

where $A_b \equiv \frac{1+\gamma(\beta\rho)}{g(\beta\rho)}$. Thus,

$$bb^* = A_b^2 \frac{VV^*}{\mu(1+\gamma^*) + \mu^*(1+\gamma)}.$$

Now substitute for bb^* into the equation (21) that defines JJ^* to obtain

$$JJ^* = NA_b^2 \frac{VV^*}{\mu(1+\gamma^*) + \mu^*(1+\gamma)} \sigma_e^2 + \sigma_u^2.$$

Next, apply Lemma A.2 to equation (24) to obtain a simplified expression for μ ,

$$\mu = J^{-1}A_\mu V,$$

where $A_\mu \equiv \frac{Nb(\beta\rho)}{J(\beta\rho)}\sigma_e^2$. Also, rewriting our solution for γ in Lemma A.1 yields

$$\gamma = -J^{-1}[J^{*-1}b^*b\sigma_e^2]_+ = -J^{-1}[J^{*-1}\frac{J^*J - \sigma_u^2}{N}]_+ = -\frac{1}{N} + J^{-1}J(0)^{-1}\frac{\sigma_u^2}{N},$$

so that

$$1 + \gamma = \frac{N-1}{N} + J^{-1}J(0)^{-1}\frac{\sigma_u^2}{N}.$$

Finally, substituting these values for μ and $1 + \gamma$ into the JJ^* equation yields

$$\begin{aligned} JJ^* &= NA_b^2 \frac{VV^*}{J^{-1}A_\mu V(\frac{N-1}{N} + J^{*-1}J(0)^{-1}\frac{\sigma_u^2}{N}) + J^{*-1}A_\mu V^*(\frac{N-1}{N} + J^{-1}J(0)^{-1}\frac{\sigma_u^2}{N})} \sigma_e^2 + \sigma_u^2 \\ &= N \frac{A_b^2}{A_\mu} \frac{VV^*JJ^*}{V(\frac{N-1}{N}J^* + J(0)^{-1}\frac{\sigma_u^2}{N}) + V^*(\frac{N-1}{N}J + J(0)^{-1}\frac{\sigma_u^2}{N})} \sigma_e^2 + \sigma_u^2. \end{aligned} \quad (27)$$

This is a non-linear functional equation in J . It is highly algebraic in character, in that if J is a rational function then the solution is determined by the roots of a polynomial of finite or possibly infinite order. To find these roots, we substitute the constants

$$b(\beta\rho) = \frac{1+\gamma(\beta\rho)}{g(\beta\rho)^2}V(\beta\rho) \quad \text{and} \quad 1 + \gamma(\beta\rho) = \frac{N-1}{N} + J(\beta\rho)^{-1}J(0)^{-1}\frac{\sigma_u^2}{N}$$

from our solutions for b and $1 + \gamma$ into the ratio A_b^2/A_μ in (27),

$$\frac{A_b^2}{A_\mu} = \left(\frac{1 + \gamma(\beta\rho)}{g(\beta\rho)} \right)^2 \frac{J(\beta\rho)}{Nb(\beta\rho)\sigma_e^2},$$

and manipulate, simplifying (27) to

$$JJ^* = \frac{1}{\frac{V^{-1}(\frac{N-1}{N}J(0)J + \frac{\sigma_u^2}{N})}{V(\beta\rho)^{-1}(\frac{N-1}{N}J(0)J(\beta\rho) + \frac{\sigma_u^2}{N})} + \frac{V^{*-1}(\frac{N-1}{N}J(0)J^* + \frac{\sigma_u^2}{N})}{V(\beta\rho)^{-1}(\frac{N-1}{N}J(0)J(\beta\rho) + \frac{\sigma_u^2}{N})}} JJ^* + \sigma_u^2. \quad (28)$$

We next make an initial conjecture that J is a k^{th} -order rational function of the form

$$J = J(0) \frac{\prod_{i=1}^k (1 - f_i z)}{\prod_{i=1}^k (1 - a_i z)},$$

with the interspersed property, $1 > a_1 > f_1 > a_2 > f_2 > \dots > a_k > f_k > 0$. We then iterate on the mapping in (28), establishing the key fact that this property is preserved by the iteration.

Substituting for V and the conjecture for J , we write the $V^{-1}(\frac{N-1}{N}J(0)J + \frac{\sigma_u^2}{N})$ term in the denominator of (28) as

$$\begin{aligned} V^{-1}(\frac{N-1}{N}J(0)J + \frac{\sigma_u^2}{N}) &= (1 - \rho z) \left(\frac{N-1}{N}J(0)^2 \frac{\prod_{i=1}^k (1 - f_i z)}{\prod_{i=1}^k (1 - a_i z)} + \frac{\sigma_u^2}{N} \right) \\ &= \frac{1 - \rho z}{\prod_{i=1}^k (1 - a_i z)} \left(\frac{N-1}{N}J(0)^2 \prod_{i=1}^k (1 - f_i z) + \prod_{i=1}^k (1 - a_i z) \frac{\sigma_u^2}{N} \right). \end{aligned} \quad (29)$$

The right-hand side of (28) then takes the form

$$\frac{1}{S(z) + S(\beta z^{-1})} J J^* + \sigma_u^2, \quad (30)$$

where $S(z)$ is implicitly defined from (29).

Increment of the polynomial order. To ease presentation, we set $\beta = 1$; generalizations to $\beta < 1$ obtain immediately with a rescaling of z and the coefficients of the Laurent expansions of the functions.¹⁰ We now show that the polynomial order in the numerator and denominator of J increases by exactly one on each iteration. To see this, inspect the mapping in (30). S includes a $V^{-1}J$ term and its conjugate counterpart has a $V^{*-1}J^*$ term: J and J^* have order- k numerators and V^{-1} and V^{*-1} have order-1 numerators, so their products are order- $(k+1)$. This numerator enters the denominator of equation (30), implying that the denominator increments to $k+1$. Finally, getting a common denominator on the right-hand side involves multiplying σ_u^2 by the order- $(k+1)$ denominator of the first term, yielding an order- $k+1$ numerator. In particular, if $a_i \neq \rho$, for all i , then canceling the common $\prod_{i=1}^k (1 - a_i z)$ in the denominator of $S(z)$ and $J(z)$, and the common $\prod_{i=1}^k (1 - a_i z^{-1})$ in the denominator of $S(z^{-1})$ and $J(z^{-1})$, and establishing a common denominator for the iterated $J J^*$ yields

$$J J^* \propto \frac{\prod_{i=1}^k (1 - f_i z) \prod_{i=1}^k (1 - f_i z^{-1}) + \sigma_u^2 \prod_{i=1}^{k+1} (1 - a'_i z) \prod_{i=1}^{k+1} (1 - a'_i z^{-1})}{\prod_{i=1}^{k+1} (1 - a'_i z) \prod_{i=1}^{k+1} (1 - a'_i z^{-1})}, \quad (31)$$

where the denominator $\prod_{i=1}^{k+1} (1 - a'_i z) \prod_{i=1}^{k+1} (1 - a'_i z^{-1})$ is obtained by (a) establishing a common denominator for the terms of $S + S^*$ that have not been canceled and then writing it in factored form, and (b) the proportionality constant is $J(0)^2$ divided by the constant generated by the factoring. If, instead, $a_i = \rho$, for some i , then it cancels with the $1 - \rho z$ in the numerator of (29), and it hence does not cancel with the $1 - \rho z$ in $J J^*$, in which case ρ continues to enter the iterated version of $J J^*$, and there are k denominator terms in $S + S^*$.

Interspersion property preserved by iteration. Next observe that if iteration preserves the interspersion structure and the iteration is contractive, then the fixed point must also have this structure. To determine how the properties of the iterated values a'_i and f'_i relate to their k^{th} -order values, a_i and f_i , first observe that when $z = a'_i$, that $\prod_{i=1}^{k+1} (1 - a'_i z^{-1}) = 0$ (or $\prod_{i, a_i \neq \rho} (1 - a'_i z^{-1}) = 0$ if $a_i = \rho$ for some i). Hence, the iterated values a'_i solve $S(z) + S(z^{-1}) = 0$. Observe that a_i cannot be a zero of the numerator of $S(z) + S(z^{-1})$: $\prod (1 - a_i z^{-1})$ appears in the denominator of $S(z^{-1})$, so that $S(a_i) + S(a_i^{-1}) = S(a_i) + \infty \neq 0$. Hence, any conjectured approximation of the equilibrium is perturbed by iteration in the sense that $a'_i \neq a_i$, for $a_i \neq \rho$.

Suppose first that $a_1 = \rho$. The largest zero of $S(z) + S(z^{-1})$ must be less than ρ , because $S(z) + S(z^{-1}) > 0$ for z between ρ and 1. Note that $\lim_{z \downarrow a_2} S(z^{-1}) = -\infty$, and $S(z^{-1})$ is continuous and monotonic on $(a_2, \rho]$. Hence, there is a unique zero in (a_2, ρ) . Similar reasoning demonstrates that the other zeros are such that $a_{i+1} < a'_{i+1} < a_i$ for $i = 1, \dots, k$, with the convention $a_{k+1} = 0$.

Now consider $a_1 \neq \rho$. First let $a_1 < \rho$. For $z > \rho$, we have $S(z) > 0$ and $S(z^{-1}) > 0$, so that $a'_1 < \rho$. For $a_2 < z < a_1$, it is also true that $S(z) > 0$ and $S(z^{-1}) > 0$: this is because there are two negative terms, $1 - \rho z^{-1}$, and $1 - a_1 z^{-1}$, whose signs cancel each other out. For $z \in (a_1, \rho)$ $S(z^{-1}) < 0$ and $S(z^{-1})$ approaches $-\infty$ as z approaches a_1 from above. Hence, $S(z) + S(z^{-1})$ must have a zero in the interval (a_1, ρ) . The same reasoning establishes the interspersion properties for the other a'_i terms.

¹⁰That is, it is innocuous to treat S^* as if it is the complex conjugate of S . See the online appendix <http://econ.uiuc.edu/~bart/frequencydomainmethods> for a detailed explanation.

Now, let $a_1, \dots, a_j > \rho$. Again, the iterated value of a_1, a'_1 , cannot exceed a_1 : all the terms $1 - a_i/a'_i$, $1 - f_i/a'_i$, $1 - a_i a'_i$, and $1 - f_i a'_i$ would then be positive. Hence, $a'_1 < a_1$. Next note that if $f_1 > \rho$, then $S(z^{-1})$ switches sign at f_1 , so that $a'_1 > f_1 > \rho$, and if $a_1 > \rho > a_2$, then $S(z^{-1})$ is negative for $\rho < z < a_1$, and becomes positive again for $z < \rho$, so that again $S(z) + S(z^{-1})$ has a crossing point at zero somewhere in the interval (ρ, a_1) , and hence $a'_1 \in (\rho, a_1)$. Again, as above, the interspersing property holds for the other a'_i terms.

Interspersing properties of the f'_i . By assumption, $a_i > f_i > a_{i+1} > 0$, and we just established that if $a_1 \geq \rho$ then $a_i > a'_{i+1} > a_{i+1}$; and if $a_1 < \rho$ then $a'_i > a_i > a'_{i+1}$. Rewriting the JJ^* mapping (31) as

$$JJ^* \propto \frac{\prod_{i=1}^k (1 - f_i z) \prod_{i=1}^k (1 - f_i z^{-1})}{\prod_{i=1}^{k+1} (1 - a'_i z) \prod_{i=1}^{k+1} (1 - a'_i z^{-1})} + \sigma_u^2,$$

we see that the first zero of the numerator, f'_1 , must have the property $a'_1 > f'_1 > a'_2$, as $\sigma_u^2 > 0$, and continuing to alternate signs, we must have $a'_2 > f'_2 > a'_3$, and so on.

Finally, because the f'_i are interspersed between the a'_i terms, applying Lemma A.5 establishes that $c_i > 0$, and conversely $c_i > 0$ implies that the f'_i are interspersed between the a'_i terms.

The set $\{a_i\}$ is infinite in equilibrium. Because the coefficients $\{a_i\}$ of the next round of iteration are strictly interspersed (except possibly for $a_i = \rho$), their order increases by one on each iteration. Hence, the set of $\{a_i\}$ cannot be finite in equilibrium.

Bounding the modulus of the mapping. Denote the mapping implicit in (28) by T^* , with $T^* : H^2(\beta) \rightarrow H^2(\beta)$, where $H^2(\beta)$ is the space of analytic square-integrable functions on the β -disk, $\beta = \{z | |z| \leq \beta^{1/2}\}$. We first show that the modulus of T^* is a fraction under the assumption that the interspersing property holds, i.e., that $1 > a_1 > f_1 > a_2 > \dots > 0$, and then verify that T^* maps $H^2(\beta)$ into itself.

Integrating (30) around the unit circle to calculate the norm yields

$$\|J^2\| = \|\nu J^2\| + \sigma_u^2, \quad (32)$$

where $\nu \nu^* \equiv \frac{1}{S + S^*}$. We establish the contraction property by showing that¹¹

$$\left\| \frac{1}{S + S^*} \right\| = \left\| \frac{1}{2\text{Re}(S)} \right\| < 1.$$

We cannot compute the norm of the denominator directly, so we use an indirect argument, bounding the infimum of the denominator from below by one, which bounds the norm of $\frac{1}{2\text{Re}(S)}$ from above by one:

$$\frac{1}{2\pi i} \oint_{\{|z|=1\}} \frac{1}{2\text{Re}(S)} \frac{dz}{z} \leq \frac{1}{\inf_{\{|z|=1\}} 2\text{Re}(S)} < 1.$$

We now bound this expression. To minimize presentation of sub-cases, we exploit the fact that on iteration $a_1 \rightarrow \rho$, and characterize that case. Recall that the structure of J is

$$J = J(0) \frac{\prod_{i=1}^k (1 - f_i z)}{\prod_{i=1}^k (1 - a_i z)},$$

with $a_i \leq \rho$ and $f_i \leq \rho$. Write the numerator of S as

$$\frac{N-1}{N} J(0)^2 \frac{\prod_{i=1}^k (1 - f_i z)}{\prod_{i=2}^k (1 - a_i z)} + \frac{\sigma_u^2}{N} (1 - \rho z),$$

¹¹We again note that we are treating S^* as if it were the complex conjugate of S via our temporary assumption that $\beta = 1$; again, see the online appendix <http://econ.uiuc.edu/~bart/frequencydomainmethods>.

where we note that because of the product $V^{-1}J$ in the numerator, there is one more f_i term than a_i term. In common denominator form, the numerator becomes

$$\frac{\frac{N-1}{N}J(0)^2 \prod_{i=1}^k (1 - f_i z) + \frac{\sigma_u^2}{N}(1 - \rho z) \prod_{i=2}^k (1 - a_i z)}{\prod_{i=2}^k (1 - a_i z)},$$

which, using a direct extension of Lemma A.4, we write as

$$C \frac{\prod_{i=1}^k (1 - \tilde{f}_i z)}{\prod_{i=2}^k (1 - a_i z)},$$

where C is a constant, and $\rho > \tilde{f}_i > f_k$. Therefore, returning to the definition of $S + S^*$ in (28) and (30), we just need to prove that

$$2 \inf_{\{|z|=1\}} \operatorname{Re} \left(\frac{\prod_{i=1}^k (1 - \tilde{f}_i z)}{\prod_{i=2}^k (1 - a_i z)} \right) > 1.$$

Because of the interspersing property, i.e., $a_i > f_i > a_{i+1}$ we can apply Lemma A.7 to this expression, which establishes that the modulus of T^* is a fraction. Therefore T^* is a contraction. Because the space of square-integrable analytic functions H^2 is complete, there is a fixed point. \square

A.5 The fixed point is a stationary linear equilibrium

We now prove that the fixed point identified by the contraction mapping is a stationary linear equilibrium. The essential step is to prove that the boundedness condition stated in Proposition 3.2 holds in equilibrium. We begin establishing this boundedness property with Lemma A.8, which characterizes the structure of the equilibrium pricing rule from the perspective of an individual speculator. After that, we prove the boundedness result in Lemma A.9. We make use there of Proposition B.1, proved in Appendix B, which is a purely technical result about the positive definiteness of a matrix, A_T , that shows up in the proof.

Lemma A.8 *If the pricing rule and trading strategies of speculators $j \neq i$ take the forms given in Proposition 4.4, then the price process as a function of i 's order flow history takes the form*

$$p_t(\mathbf{x}_{it}, \bar{\xi}_t) = \bar{\xi}_t + \hat{\xi} [\xi_0 x_{it} + \xi_1 x_{it-1} + \xi_2 x_{it-2} + \dots].$$

$\bar{\xi}_t$ is a random term reflecting stochastic components of trades by speculators $j \neq i$ and noise traders that decays faster than ρ . Further, $\xi_0 > \xi_1 > \dots > 0$, and $\hat{\xi} > 0$.

Proof: From i 's perspective, $p_t(\mathbf{x}_{it}) = q(L)\lambda(L)x_{it}$. Substituting for

$$q(L) = \frac{1}{1 - (N-1)B(L)}, \quad \gamma(L) = \frac{B(L)}{1 - NB(L)} \quad \text{and} \quad \Gamma(L) = 1 + N\gamma(L) = \frac{1}{1 - NB(L)}, \quad (33)$$

and defining $\mu(L) = \Gamma(L)\lambda(L)$, we solve for

$$q(L)\lambda(L) = (1 + \gamma(L))^{-1}\Gamma(L)\lambda(L) = (1 + \gamma(L))^{-1}\mu(L). \quad (34)$$

Algebraic manipulation then reveals that

$$1 + \gamma(L) = \frac{1 - (N-1)B(L)}{1 - NB(L)} = q(L)^{-1}\Gamma(L).$$

Recall that the finite approximation of b takes the multiplicative form

$$b(L) = c_0 \frac{\prod_{i=1}^{k-1} (1 - c_i L)}{\prod_{i=1}^k (1 - a_i L)}.$$

$J(L)$ takes a similar form, save that the polynomial orders of its numerator and denominator are equal:

$$J(L) = J(0) \frac{\prod_{i=1}^k (1 - f_i L)}{\prod_{i=1}^k (1 - a_i L)}. \quad (35)$$

We now characterize the structure of $\gamma(L)$ and $\mu(L)$, recalling that

$$\gamma = J^{-1}[J^{*-1}b^*b\sigma_e^2]_+ \quad \text{and} \quad \mu = J^{-1}[J^{*-1}Nb^*V\sigma_e^2]_+.$$

Recall that b is the sum of AR terms. From Lemma A.2, the expression for γ , $[J^{*-1}b^*b\sigma_e^2]_+$ is also the sum of AR(1) terms, i.e., the annihilate preserves the AR structure of b . Similarly, $[J^{*-1}Nb^*V\sigma_e^2]_+$ is an AR(1). To characterize γ and μ , we use this structure to determine the polynomial orders of the numerators and denominators of $1 + \gamma$ and of μ . Finding the common denominator of $1 + \gamma(L)$ we have

$$1 + \gamma(L) \sim \frac{\prod_{i=1}^k (1 - g_i L)}{\prod_{i=1}^k (1 - f_i L)}, \quad (36)$$

where we ignore the leading multiplicative constant. Similarly,

$$\mu(L) \sim \frac{\prod_{i=1}^{k-1} (1 - a_i L)}{\prod_{i=1}^k (1 - h_i L)},$$

where the order of the numerator is only $k - 1$ because $V(L)$ cancels the first AR term in $J(L)$. Because the polynomial orders are such that the product $(1 + \gamma(L))^{-1}\mu(L)$ has numerator polynomial order that is less than the polynomial order of the denominator, it can be expressed in partial-fractions form.

The autoregressive coefficients of $\mu(L)$ lie in $(0, 1)$. Hence, examining (34), it suffices to show that the autoregressive coefficients of $(1 + \gamma(L))^{-1}$ also lie in $(0, 1)$, i.e., the g_i in (36) lie in $(0, 1)$. We have

$$\gamma = J^{-1}[J^{*-1}b^*b\sigma_e^2]_+ = J^{-1}[J^{*-1}\frac{J^*J - \sigma_u^2}{N}]_+ = \frac{1}{N}(1 - J^{-1}[J^{*-1}\sigma_u^2]_+).$$

Substituting the rational function structure from (35) for J yields

$$(1 + \gamma) \sim \frac{\frac{N-1}{N} \prod_{i=1}^k (1 - f_i L) + J(0)^{-2} \prod_{i=1}^k (1 - a_i L) \sigma_u^2}{\prod_{i=1}^k (1 - f_i L)}.$$

The numerator term of $(1 + \gamma)$ is a convex combination of two terms with coefficients $a_i \in (0, 1)$ and $f_i \in (0, 1)$, so that the implied coefficients of the convex combination, the g_i , also have the property $g_i \in (0, 1)$. It follows that ξ_k is formed from the sum of elements of a geometric series that arises from expanding each AR term, and $\xi_0 > \xi_1 > \dots > 0$ as claimed.

It remains to prove that $\hat{\xi} > 0$. To do this, we show that the leading coefficients of the constituent expressions for price, $(1 + \gamma(L))^{-1}$, and $\mu(L)$ are positive. We first show that $\|\gamma\| \leq \frac{1}{N}$:

$$\begin{aligned} \|\gamma\| &= \|J^{-1}[J^{*-1}b^*b]_+ \sigma_e^2\| \leq \|J^{-1}[J^{*-1}\frac{J^*J - \sigma_u^2}{N}]_+\| \\ &\leq \frac{1}{N} \|1 - J^{-1}[J^{*-1}\sigma_u^2]_+\| \leq \frac{1}{N} \left(1 - \inf_z |J^{-1}|^2 \sigma_u^2\right) \leq \frac{1}{N}. \end{aligned}$$

It follows that the leading coefficient of $1 + \gamma(L)$ and hence of $(1 + \gamma(L))^{-1}$ is positive. Turning to the leading coefficient of $\mu(L)$, recall that

$$\mu = J^{-1}[J^{*-1}Nb^*V\sigma_e^2]_+.$$

Note that the $J(L)$ function is formed from a factorization. Hence, $J(0)$ is a standard deviation, which is positive, and therefore the leading term of J^{-1} is positive. By assumption, the leading coefficient of $V(L)$ is unity.

Finally, from the definition of J , we have

$$bb^* = \frac{JJ^* - \sigma_u^2}{N\sigma_e^2}.$$

Because $J(L)$ is formed from a factorization, $JJ^* - \sigma_u^2$ is a Hermitian positive definite rational function. As such, it can be factored, implying that the leading coefficient of $b(L)$ is positive. \square

Lemma A.9 *Suppose the pricing rule and trading strategies of speculators $j \neq i$ take the forms given in Proposition 4.4. Write $p_{t+s}(\mathbf{x}_{it+s})$ to be the price, incorporating the fixed stationary linear responses by the other speculators, as a function of speculator i 's current and lagged orders. Let $\{x_{it+s}\}_{s=0}^\infty$ maximize i 's discounted expected profits,*

$$E_t \left[\sum_{s=0}^\infty \beta^s (v_{t+s} - p_{t+s}(\mathbf{x}_{it+s})) x_{it+s} \mid \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right],$$

where we omit the formal dependence of i 's trades on his information. Take $\epsilon > 0$. Then there exists an $s(\mathbf{e}_{it}, \boldsymbol{\Omega}_t, \epsilon)$ such that for all $\tau > s(\mathbf{e}_{it}, \boldsymbol{\Omega}_t, \epsilon)$

$$E_t \left[\sum_{s=\tau}^\infty \beta^s (v_{t+s} - p_{t+s}(x_{it+s})) x_{it+s} \mid \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right] < \epsilon.$$

Proof: The proof strategy is the following. We first decompose speculator i 's expected profit function given time t information into two components, a “deterministic component” that is a function of time t information, and a “stochastic component” that reflects expected future trading profits stemming from the yet-to-be realized future signals and noise trades. We next show that the stochastic component of profits is a concave function of speculator i 's orders and has a bounded expected payoff. To do this, we find a gross upper bound on these expected payoffs, by calculating speculator i 's profits when he knows all future realizations of future signals and noise trades—showing that his optimal strategy is a linear function of this information, and computing the resulting profits—and then integrating over the possible signals and noise trades. This stochastic component retains its structure in future periods, and by an appropriate choice of $t + s$ can be made arbitrarily small in terms of date t value due to discounting.

Speculator i 's objective can be decomposed into

$$\begin{aligned} \max_{\{x_{it+s}\}} E_t & \left[\sum_{s=0}^\infty \beta^s [\rho^s v_t - \bar{\zeta}_{t+s}] x_{it+s} - \left[\hat{\xi} [\xi_0 x_{it+s} + \xi_1 x_{it+s-1} + \xi_2 x_{it+s-2} + \dots] \right] x_{it+s} \mid \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right] \\ & + E_t \left[\sum_{s=0}^\infty \beta^s [v_{t+s} - \rho^s v_t - (\bar{\xi}_{t+s} - \bar{\zeta}_{t+s})] x_{it+s} - \left[\hat{\xi} [\xi_0 x_{it+s} + \xi_1 x_{it+s-1} + \xi_2 x_{it+s-2} + \dots] \right] x_{it+s} \mid \mathbf{e}_{it}, \boldsymbol{\Omega}_t \right], \end{aligned}$$

where $\bar{\zeta}_{t+s}$ is the portion of $\bar{\xi}_{t+s}$ that is forecastable by speculator i given his date- t information. Letting $w_{t+s} = v_{t+s} - \rho^s v_t - (\bar{\xi}_{t+s} - \bar{\zeta}_{t+s})$, a strict upper bound on the expected payoff associated with the second

sum in i 's objective is obtained when speculator i optimizes given knowledge of the entire future history of $\{w_{t+s}\}_{s \geq t}$, and we then integrate over all possible future histories. So consider

$$\int_w \left[\max_{\{x_{it+s}\}_{s \geq 0}} \sum_{s=0}^{\infty} \beta^s w_{t+s} x_{it+s} - \left[\hat{\xi} [\xi_0 x_{it+s} + \xi_1 x_{it+s-1} + \xi_2 x_{it+s-2} + \dots] \right] x_{it+s} | \{w_{t+s}\} \right]$$

For any realized future path of the $\{w_{t+s}\}$, the speculator solves the deterministic problem

$$\max_{\{x_{it+s}\}_{s \geq 0}} \left(\sum_{s=0}^{\infty} \beta^s w_{t+s} x_{it+s} - \left[\hat{\xi} [\xi_0 x_{it+s} + \xi_1 x_{it+s-1} + \xi_2 x_{it+s-2} + \dots] \right] x_{it+s} \right)$$

$$x_{i0}, \dots, x_{it} \quad \text{given.}$$

First consider the finite-horizon version of this deterministic problem.

$$\max_{\{x_{it}, \dots, x_{it+T}\}} \begin{pmatrix} x_{it} & x_{it+1} & \dots & x_{it+s} & \dots & x_{it+T} \end{pmatrix} \begin{pmatrix} w_t \\ \beta w_{t+1} \\ \vdots \\ \beta^s w_{t+s} \\ \vdots \\ \beta^T w_{t+T} \end{pmatrix}$$

$$- \begin{pmatrix} x_{it} & x_{it+1} & \dots & x_{it+s} & \dots & x_{it+T} \end{pmatrix} \begin{pmatrix} \xi_t & \xi_{t-1} & \dots & \xi_0 & 0 & 0 & \dots & 0 \\ \beta \xi_{t+1} & \beta \xi_t & \dots & \beta \xi_1 & \beta \xi_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta^s \xi_{t+s} & \beta^s \xi_{t+s-1} & \dots & \beta^s \xi_t & \beta^s \xi_{t-1} & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta^T \xi_{t+T} & \beta^T \xi_{t+T-1} & \dots & \beta^T \xi_{t+T-s} & \beta^T \xi_{t+T-s-1} & \dots & \dots & \beta^T \xi_0 \end{pmatrix} \begin{pmatrix} x_{i0} \\ x_{i1} \\ \vdots \\ x_{it+s} \\ \vdots \\ x_{it+T} \end{pmatrix}.$$

We write this problem more compactly as

$$\max_{x_T} x'_T w_T - x'_T \begin{pmatrix} N_T & M_T \end{pmatrix} \begin{pmatrix} x^t \\ x_T \end{pmatrix},$$

where

$$M_T \equiv \begin{pmatrix} \xi_0 & 0 & 0 & \dots & 0 \\ \beta \xi_1 & \beta \xi_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta^s \xi_t & \beta^s \xi_{t-1} & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta^T \xi_{t+T-s} & \beta^T \xi_{t+T-s-1} & \dots & \dots & \beta^T \xi_0 \end{pmatrix},$$

and the time subscripts emphasize that we are considering the finite-horizon problem. The problem consists of a linear part, $x'_T w_T - x'_T N_T x^t$, and a quadratic part, $x'_T M_T x_T$. We calculate the second-order condition of the finite-horizon problem explicitly, determining that the central matrix M_T is negative definite. Removing the negative sign, this is equivalent to establishing that the internal matrix is positive definite. We establish negative definiteness by considering the symmetrized version of the problem:

$$\max_{x_T} x'_T w_T - \frac{1}{2} x'_T (M_T + M'_T) x_T = \max_{x_T} x'_T w_T - x'_T A_T x_T.$$

Because A_T is symmetric, A_T is positive definite if its eigenvalues are positive. Lemma A.8 established that the price impact of i 's trades is given by the sum of AR(1) expressions with $\xi_0 > \xi_1 > \dots > 0$. Lemma B.1 establishes the positive definiteness of A_T by expanding it into the infinite sum of matrices, each associated with one of the AR(1) expressions, which we show are positive definite. Because the sum of positive definite matrices is positive definite, it follows that A_T is positive definite.

Finally, we must address the fact that in the limit, the final row of the matrix M_T , which is multiplied by β^T , converges to zero, so that the strict negative definiteness of $-A_\infty$ might fail. We must show that there is no strategy that can exploit this asymptotic linearity. To drive the quadratic penalty to zero, a necessity for the arbitrage strategy, the growth of $|x_{it+s}|$ must be bounded below $1/\beta^s$ so that the shrinkage of β^s dominates it. But the gain from this strategy occurs from the linear term in the objective, and this is discounted at rate β^s . Therefore, the gain from this strategy shrinks to zero, and it must be suboptimal. Hence, speculator i 's optimal trading strategy in this perfect foresight economy is characterized by the linear first-order conditions, and quadratic payoffs. Integrating over all possible signal realizations and all possible noise trades, we find that the expected payoff from the ‘‘stochastic component’’ of the objective is bounded.

It remains to show that the future contribution of the deterministic component of profits given date t information can be made arbitrarily small. Due to discounting and the autoregressive decay in the contributions of private information to the price process, the future contribution to profits of speculator i 's date t private information (including information in prices) can be made arbitrarily small by a choice of a future date $t + s(\mathbf{e}_{it}, \mathbf{\Omega}_t, \epsilon)$, so that for all $s > s(\mathbf{e}_{it}, \mathbf{\Omega}_t, \epsilon)$,

$$\beta^s [E[v_{it+s}|\mathbf{e}_{it}, \mathbf{\Omega}_{it}] - E[\bar{\xi}_{t+s}|\mathbf{e}_{it}, \mathbf{\Omega}_{it}]]$$

is arbitrarily small.

It follows that the only way to make sufficiently distant future profits non-trivial is to adopt a trading strategy that induces increasingly large deviations between price and value. However, the structure of pricing implies that this requires trade sizes to grow at a rate exceeding $\frac{1}{\beta}$, and the quadratic convex structure of the price impact of trades that we have established then implies that this generates arbitrarily high negative profits, a contradiction of optimality. \square

Proof: (Proposition 4.5) This follows immediately from Proposition 4.4 and the above lemmas. \square

A.6 Proofs of characterization results

Lemma A.10 *The dynamic structure of J is determined entirely by the structure of V and N , and is independent of σ_e^2 and σ_u^2 .*

Proof: We make the mapping in (28) dimensionless by dividing by σ_u^2/N :

$$\bar{J}\bar{J}^* = \frac{1}{\frac{V^{*-1}((N-1)\bar{J}(0)\bar{J}^*+1)}{V(\beta\rho)^{-1}((N-1)\bar{J}(0)\bar{J}(\beta\rho)+1)} + \frac{V^{-1}((N-1)\bar{J}(0)\bar{J}+1)}{V(\beta\rho)^{-1}((N-1)\bar{J}(0)\bar{J}(\beta\rho)+1)}} \bar{J}\bar{J}^* + 1,$$

which is independent of σ_e^2 and σ_u^2 . \square

Lemma A.11 $J \propto \sigma_u$ and $b \propto \frac{\sigma_u}{\sigma_e}$.

Proof: The result for J follows immediately from inspecting the JJ^* mapping in (28). The result for b follows from the identity

$$bb^* = \frac{JJ^* - \sigma_u^2}{N\sigma_e^2}. \quad \square$$

Proof: (Proposition 4.6) This is a direct result of factoring in the definition of J from JJ^* in (21) and finding the partial-fractions representation. \square

Proof: (Proposition 4.7) This follows directly from the equations defining μ and γ . \square

Proof: (Proposition 4.8) Substituting for $\mu(z) = J^{-1}[J^{*-1}Nb^*\sigma_e^2V]_+$ into the price process, $\mu(z)J(z)w_t$, yields $[J^{*-1}Nb^*\sigma_e^2V]_+$. By Lemma A.2 the annihilate of this product of non-analytic functions with the AR(1) process $V(z)$ is equal to

$$J^{-1}(\beta\rho)Nb(\beta\rho)\sigma_e^2V(L),$$

where the constant is $J^{-1}(\beta\rho)Nb(\beta\rho)\sigma_e^2$, and we recall that w_t is a white noise process. \square

Proof: (Propositions 5.1 and 5.2) We proved in Lemma A.11 that $J \propto \sigma_u$ and $b \propto \frac{\sigma_u}{\sigma_e}$. Hence,

$$\gamma(z) \propto \frac{b(0)^2}{J(0)^2}\sigma_e^2 \propto \frac{\frac{\sigma_u^2}{\sigma_e^2}}{\sigma_u^2}\sigma_e^2 \propto 1.$$

Therefore, $1 + N\gamma \propto 1$. Now recall that

$$\mu(z) = J^{-1}[J^{*-1}Nb^*\sigma_e^2V]_+ \propto \frac{\sigma_e^2}{\sigma_u^2}[b^*V]_+ \propto \frac{\sigma_e^2}{\sigma_u^2}\frac{\sigma_u}{\sigma_e}V \propto \frac{\sigma_e}{\sigma_u}.$$

Recall that the pricing filter $\lambda = \mu/(1 + N\gamma)$. Therefore, $\lambda \propto \frac{\sigma_e}{\sigma_u}$. The result for profit is similar. From the objective (7), an informed trader's expected profit is:

$$\pi = N(V - \mu b)\gamma b\sigma_e^2 + (V - \mu b)b\sigma_e^2 + \mu\sigma_u^2.$$

Using the proportionality results for b , μ , and γ , we have

$$\mu b \propto 1; \quad \gamma b\sigma_e^2 \propto \sigma_u\sigma_e; \quad b\sigma_e^2 \propto \sigma_u\sigma_e; \quad \mu\sigma_u^2 \propto \gamma\sigma_e;$$

and therefore $\pi \propto \sigma_e\sigma_u$. \square

B A technical lemma on the positive definiteness of a matrix

Lemma B.1 *Let $0 < \beta < 1$, $0 < a < 1$, and define a representative term of the sum,*

$$W_T \equiv \frac{1}{2} \left(\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ \beta a & \beta & 0 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ \beta^\tau a^\tau & \beta^\tau a^{\tau-1} & \dots & \beta^\tau & \dots & 0 \\ \vdots & & & & & \vdots \\ \beta^T a^T & \beta^T a^{T-1} & \beta^T a^{T-2} & \dots & \dots & \beta^T \end{pmatrix} + \begin{pmatrix} 1 & \beta a & \dots & \beta^\tau a^\tau & \dots & \beta^T a^T \\ 0 & \beta & \dots & \beta^\tau a^{\tau-1} & \dots & \beta^T a^{T-1} \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & \beta^\tau & \dots & \beta^T a^{T-\tau} \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & 0 & \dots & \beta^T \end{pmatrix} \right).$$

Then W_T is positive definite.

PROOF: Write W_T explicitly as

$$\begin{aligned} W_T &= \begin{pmatrix} 2 & \beta a & \beta^2 a^2 & \beta^3 a^3 & \dots & \beta^T a^T \\ \beta a & 2\beta & \beta^2 a & \beta^3 a^2 & \dots & \beta^T a^{T-1} \\ \vdots & & & & & \vdots \\ \beta^\tau a^\tau & \beta^\tau a^{\tau-1} & \dots & 2\beta^\tau & \dots & \beta^T a^{T-\tau} \\ \vdots & & & & & \vdots \\ \beta^T a^T & \beta^T a^{T-1} & \beta^T a^{T-2} & \dots & \dots & 2\beta^T \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \beta & 0 & \dots & 0 \\ \vdots & & \vdots & & \\ 0 & 0 & 0 & \dots & \beta^T \end{pmatrix} + \begin{pmatrix} 1 & \beta a & \beta^2 a^2 & \beta^3 a^3 & \dots & \beta^T a^T \\ \beta a & \beta & \beta^2 a & \beta^3 a^2 & \dots & \beta^T a^{T-1} \\ \vdots & & & & & \vdots \\ \beta^\tau a^\tau & \beta^\tau a^{\tau-1} & \dots & \beta^\tau & \dots & \beta^T a^{T-\tau} \\ \vdots & & & & & \vdots \\ \beta^T a^T & \beta^T a^{T-1} & \beta^T a^{T-2} & \dots & \dots & \beta^T \end{pmatrix} \\ &\equiv \Delta(\beta, T) + B_T. \end{aligned}$$

Clearly, $\Delta(\beta, T)$ is positive definite. We now establish that B_T is positive definite. B_T is positive definite if B_T^{-1} is positive definite, and B_T^{-1} is positive definite if its eigenvalues are all strictly positive because B_T^{-1} is

symmetric. One can verify by direct multiplication that

$$\begin{aligned}
B_T^{-1} &= \frac{1}{1-\beta a^2} \begin{pmatrix} 1 & -a & 0 & 0 & \dots & 0 & 0 & 0 \\ -a & \frac{1+\beta a^2}{\beta} & -\frac{a}{\beta} & 0 & \dots & 0 & 0 & 0 \\ 0 & -\frac{a}{\beta} & \frac{1+\beta a^2}{\beta^2} & -\frac{a}{\beta^2} & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\frac{a}{\beta^{T-2}} & \frac{1+\beta a^2}{\beta^{T-1}} & -\frac{a}{\beta^{T-1}} \\ 0 & 0 & 0 & 0 & \dots & 0 & -\frac{a}{\beta^{T-1}} & \frac{1}{\beta^T} \end{pmatrix} \\
&= \frac{1}{1-\beta a^2} \Delta(\beta, T)^{-1} \begin{pmatrix} 1 & -a & 0 & 0 & \dots & 0 & 0 & 0 \\ -\beta a & 1+\beta a^2 & -a & 0 & \dots & 0 & 0 & 0 \\ 0 & -\beta a & 1+\beta a^2 & -a & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\beta a & 1+\beta a^2 & -a \\ 0 & 0 & 0 & 0 & \dots & 0 & -\beta a & 1 \end{pmatrix} \\
&\equiv \frac{1}{1-\beta a^2} \Delta(\beta, T)^{-1} \tilde{B}_T^{-1}.
\end{aligned}$$

The eigenvalues of B_T^{-1} are positive if the eigenvalues of \tilde{B}_T^{-1} are positive. The characteristic equation is

$$\left| \begin{pmatrix} 1-\lambda & -a & 0 & 0 & \dots & 0 & 0 & 0 \\ -\beta a & 1+\beta a^2-\lambda & -a & 0 & \dots & 0 & 0 & 0 \\ 0 & -\beta a & 1+\beta a^2-\lambda & -a & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\beta a & 1+\beta a^2-\lambda & -a \\ 0 & 0 & 0 & 0 & \dots & 0 & -\beta a & 1-\lambda \end{pmatrix} \right| = 0.$$

Define the determinant

$$D_T \equiv \left| \begin{pmatrix} 1+\beta a^2-\lambda & -a & 0 & 0 & \dots & 0 & 0 & 0 \\ -\beta a & 1+\beta a^2-\lambda & -a & 0 & \dots & 0 & 0 & 0 \\ 0 & -\beta a & 1+\beta a^2-\lambda & -a & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\beta a & 1+\beta a^2-\lambda & -a \\ 0 & 0 & 0 & 0 & \dots & 0 & -\beta a & 1-\lambda \end{pmatrix} \right|.$$

Then the characteristic equation can be written more succinctly as

$$\begin{aligned}
(1-\lambda)D_{T-1} + a &\left| \begin{pmatrix} -\beta a & -a & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1+\beta a^2-\lambda & -a & 0 & \dots & 0 & 0 & 0 \\ 0 & -\beta a & 1+\beta a^2-\lambda & -a & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -a & 1+\beta a^2-\lambda & -a \\ 0 & 0 & 0 & 0 & \dots & 0 & -\beta a & 1-\lambda \end{pmatrix} \right| \\
&= (1-\lambda)D_{T-1} - \beta a^2 D_{T-2}.
\end{aligned}$$

Thus, the characteristic equation is

$$(1 - \lambda)D_{T-1} - \beta a^2 D_{T-2} = 0.$$

It is also immediate that

$$D_t = (1 + \beta a^2 - \lambda)D_{t-1} - \beta a^2 D_{t-2}.$$

This is a second-order linear difference equation. It helps to express it in first-order nonlinear form:

$$\frac{D_t}{D_{t-1}} = (1 + \beta a^2 - \lambda) - \frac{\beta a^2}{\frac{D_{t-1}}{D_{t-2}}}$$

or

$$x_t = (1 + \beta a^2 - \lambda) - \frac{\beta a^2}{x_{t-1}}.$$

The terminal condition is then

$$\frac{D_{T-1}}{D_{T-2}} = x_{T-1} = \frac{\beta a^2}{1 - \lambda}.$$

and the initial conditions are:

$$D_1 = 1 - \lambda \quad \text{and} \quad D_2 = (1 + \beta a^2 - \lambda)(1 - \lambda) - \beta a^2,$$

so that

$$x_2 = \frac{(1 + \beta a^2 - \lambda)(1 - \lambda) - \beta a^2}{1 - \lambda}.$$

We now demonstrate that λ cannot be negative. Suppose $\lambda = -b$, $b > 0$. Then

$$x_2 = 1 + \beta a^2 + b - \frac{\beta a^2}{1 + b} > 1.$$

Similarly,

$$x_t = 1 + \beta a^2 + b - \frac{\beta a^2}{x_{t-1}} > 1,$$

if $x_{t-1} > 1$. Therefore, $x_{T-1} > 1$. But the terminal condition is then

$$x_{T-1} = \frac{\beta a^2}{1 + b} < 1,$$

which contradicts $x_{T-1} > 1$ from the previous step. Thus, \tilde{B}_T^{-1} is positive definite.

Now observe that the product $\Delta(\beta, T)^{-1} \tilde{B}_T^{-1}$ must also be positive definite. This follows because $\Delta(\beta, T)$ is diagonal. When we compute the t^{th} principle minor of the product $\Delta(\beta, T)^{-1} \tilde{B}_T^{-1}$, the minor is the determinant of the product $\Delta(\beta, t)^{-1}$ and the t^{th} principle minor of \tilde{B}_T^{-1} , which is guaranteed to be positive by the above argument. Therefore, the determinant of the product is also positive.

Now we can write

$$W_T = \Delta(\beta, T)(I + (1 - \beta a^2) \tilde{B}_T).$$

Because the sum of positive definite matrices is positive definite, this is also positive definite. \square